Math 3208 The Derivative and Tangent Lines

A. Calculate the slope of the tangent line to each of the following curves at the given value.

1. $f(x) = 4x^3 - 1$ at $x = 1$

$$f'(x) = \lim_{h \to 0} \frac{[4(x+h)^3 - 1] - [4x^3 - 1]}{h} = \lim_{h \to 0} \frac{4[(x+h)^3 - x^3]}{h}$$

$$= \lim_{h \to 0} \frac{4(\cancel{x+h-x})[(x+h)^2 + (x+h).x + x^2]}{\cancel{h}}$$

$$= 4(x^2 + x^2 + x^2) = 4.3x^2 = 12x^2$$

$$f'(x) = 12x^2$$

$$f'(1) = 12.(1)^2 = 12$$ (Slope of the tangent line at $x=1$)

2. $f(x) = x^2 + 3x - 1$ at $x = 2$

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^2 + 3(x+h) - 1] - (x^2 + 3x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2 + 3h}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-x)(x+h+x) + 3h}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+h) + 3h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h+3)}{\cancel{h}}$$

$$= 2x+3$$

$$f'(x) = 2x+3$$

$$f'(2) = 2.2 + 3 = 7$$

3. $f(x) = \sqrt{2x}$ at $x = 1$

$$\lim_{h \to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \cdot \frac{(\sqrt{2(x+h)} + \sqrt{2x})}{(\sqrt{2(x+h)} + \sqrt{2x})}$$

$$= \lim_{h \to 0} \frac{2(x+h) - 2x}{h(\sqrt{2(x+h)} + \sqrt{2x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{2(x+h)} + \sqrt{2x})}$$

$$= \frac{1}{\sqrt{2x} + \sqrt{2x}} = \frac{1}{2\sqrt{2x}}$$

$$f'(x) = \frac{1}{2\sqrt{2x}} \Rightarrow f'(1) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

4. $f(x) = \dfrac{4x+1}{x-2}$ at $x = 3$

$$\lim_{h \to 0} \frac{\frac{4(x+h)+1}{(x+h)-2} - \frac{4x+1}{x-2}}{h}$$

$$= \lim_{h \to 0} \frac{(4x+4h+1)(x-2) - (4x+1)(x+h-2)}{h(x-2)(x+h-2)}$$

$$= \lim_{h \to 0} \frac{4x^2 + 4hx + x - 8x - 8h - 2 - 4x^2 - 4xh + 8x - x - h + 2}{h(x-2)(x+h-2)}$$

$$= \lim_{h \to 0} \frac{-9\cancel{h}}{\cancel{h}(x-2)(x+h-2)} = \boxed{\frac{-9}{(x-2)^2} = f'(x)}$$

$$f'(3) = \frac{-9}{(3-2)^2} = -9$$

1. Use the definition of the derivative to determine the derivative of the following:

(a) $f(x) = 2x^2 + 3x + 5$

$$f'(x) = \lim_{h\to 0} \frac{\left[2(x+h)^2 + 3(x+h) + 5\right] - (2x^2+3x+5)}{h}$$

$$= \lim_{h\to 0} \frac{2\left[(x+h)^2 - x^2\right] + 3(x+h-x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h-x)(x+h+x) + 3h}{h}$$

$$= \lim_{h\to 0} \frac{2h(2x+h) + 3h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left[(4x+2h)+3\right]}{\cancel{h}}$$

$$\Rightarrow f'(x) = 4x+3$$

(b) $f(x) = x^3 - 12x$

$$f'(x) = \lim_{h\to 0} \frac{\left[(x+h)^3 - 12(x+h)\right] - (x^3 - 12x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3 - 12(x+h-x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h-x)\left[(x+h)^2 + (x+h).x + x^2\right] - 12h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}.\left[(x+h)^2 + (x+h).x + x^2 - 12\right]}{\cancel{h}}$$

$$= \lim_{h\to 0} \left[(x+h)^2 + (x+h).x + x^2 - 12\right]$$

$$\Rightarrow f'(x) = x^2 + x^2 + x^2 - 12 = 3x^2 - 12$$

(c) $f(x) = \dfrac{1}{2-x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{2-(x+h)} - \frac{1}{2-x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2-x-h} - \frac{1}{2-x}}{h}$$

$$= \lim_{h\to 0} \frac{2-x-(2-x-h)}{h.(2-x-h)(2-x)}$$

$$= \lim_{h\to 0} \frac{2-x-2+x+h}{h(2-x-h)(2-x)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(2-x-h)(2-x)} \Rightarrow f'(x) = \frac{1}{(2-x)^2}$$

(d) $f(x) = \sqrt{x+4}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} . \frac{(\sqrt{x+h+4} + \sqrt{x+4})}{(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \lim_{h\to 0} \frac{(x+h+4) - (x+4)}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$\Rightarrow f'(x) = \frac{1}{2\sqrt{x+4}}$$

(e) $f(x)=\frac{1}{2}x-\frac{1}{3}$

$$f'(x)=\lim_{h\to 0}\frac{\left(\frac{x+h}{2}-\frac{1}{3}\right)-\left(\frac{x}{2}-\frac{1}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x+h}{2}-\frac{x}{2}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{h}{2}}{h}=\frac{1}{2}$$

$$\Rightarrow f'(x)=\frac{1}{2}$$

(f) $f(x)=\frac{1-2x}{3+x}$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1-2(x+h)}{3+(x+h)}-\frac{1-2x}{3+x}}{h}$$

$$=\lim_{h\to 0}\frac{(1-2x-2h)(3+x)-(1-2x)(3+x+h)}{h(3+x+h)(3+x)}$$

$$=\lim_{h\to 0}\frac{3-6x-6h+x-2x^2-2hx-3-x-h+6x+2x^2+2xh}{h(3+x+h)(3+x)}$$

$$=\lim_{h\to 0}\frac{-7\cancel{h}}{\cancel{h}(3+x+h)(3+x)}\Rightarrow f'(x)=\frac{-7}{(x+3)^2}$$

(g) $f(x)=\frac{1}{10x}+x$

$$f'(x)=\lim_{h\to 0}\frac{\left[\frac{1}{10(x+h)}+(x+h)\right]-\left(\frac{1}{10x}+x\right)}{h}=\lim_{h\to 0}\frac{\frac{1}{10(x+h)}-\frac{1}{10x}+h}{h}=\lim_{h\to 0}\frac{\frac{x-(x+h)}{10x(x+h)}+h}{h}$$

$$=\lim_{h\to 0}\frac{\frac{-h}{10x(x+h)}+h}{h}=\lim_{h\to 0}\frac{\cancel{h}\left(\frac{-1}{10x(x+h)}+1\right)}{\cancel{h}}\Rightarrow \boxed{f'(x)=\frac{-1}{10x^2}+1}$$

$\boxed{y=\sqrt{x}\Rightarrow y'=\frac{1}{2\sqrt{x}}}$

2. Use the definition of the derivative to show that the derivative of $y=\frac{1}{\sqrt{x}}$ is $y'=\frac{1}{2\sqrt{x}}$.

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})}$$

$$=\lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=\lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\Rightarrow f'(x)=\frac{-1}{\sqrt{x}.\sqrt{x}(\sqrt{x}+\sqrt{x})}=\boxed{\frac{-1}{2x\sqrt{x}}}$$

3. Determine the equation of the tangent and normal lines to the curve $f(x)=\sqrt{x-4}$ at the point

x = +8

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h-4}-\sqrt{x-4}}{h}=\lim_{h\to 0}\frac{(x+h-4)-(x-4)}{h(\sqrt{x+h-4}+\sqrt{x-4})}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h-4}+\sqrt{x-4})}$$

$\Rightarrow f'(x)=\frac{1}{2\sqrt{x-4}}\Rightarrow f'(8)=\frac{1}{2\sqrt{8-4}}=\frac{1}{4}$ (Slope of the Tangent Line)

$f(8)=\sqrt{8-4}\Rightarrow f(8)=2\Rightarrow$ Tangent Line is passing through the point $(8,2)$ on the curve. (x_0,y_0)

$\circledast$ $\boxed{m.m'=-1}$ (Slopes of perpendicular Lines)

$y-y_0=m(x-x_0)$

$y-2=\frac{1}{4}(x-8)$

$\boxed{y=\frac{x}{4}}$

Tangent Line

$y-y_0=m'(x-x_0)$

$y-2=-4(x-8)$

$\boxed{y=-4x+34}$

Normal Line

Definition of Derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Use the definition of the derivative to determine the derivative of the following:

1. $f(x) = x^2 + 2x$

$$f'(x) = \lim_{h\to 0}\frac{[(x+h)^2 + 2(x+h)] - [x^2+2x]}{h}$$

$$= \lim_{h\to 0}\frac{[\cancel{x^2} + 2xh + h^2 \cancel{+2x} + 2h] - [\cancel{x^2} \cancel{+2x}]}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(2x+h+2)}{\cancel{h}}$$

$$= 2x+2$$

$$f'(x) = 2x+2 \blacksquare$$

2. $f(x) = \sqrt{x-1}$

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{x+h-1} - \sqrt{x-1})}{h}\frac{(\sqrt{x+h-1} + \sqrt{x-1})}{(\sqrt{x+h-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0}\frac{(\cancel{x}+h\cancel{-1}) - (\cancel{x}\cancel{-1})}{h(\sqrt{x+h-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h-1} + \sqrt{x-1})}$$

$$= \frac{1}{2\sqrt{x-1}} \blacksquare$$

B. Find the equation of the tangent line to each of the following curves at the given point.

1. $f(x) = x^2 - 1$ at $(2,3)$

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^2 - 1] - [x^2 - 1]}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2 \cancel{-1}) - (\cancel{x^2} \cancel{-1})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$f'(x) = 2x$$

$$\therefore f'(2) = 2(2) = 4$$

$$\therefore (y-3) = 4(x-2)$$

2. $f(x) = x^3 + 3x$ at $(1,4)$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 + 3(x+h) - [x^3 + 3x]}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{3x} + 3h] - [\cancel{x^3} + \cancel{3x}]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}[3x^2 + 3xh + h^2 + 3]}{\cancel{h}}$$

$$= 3x^2 + 3$$

$$f'(1) = 3(1)^2 + 3 = 6$$

$$\therefore (y-4) = 6(x-1)$$

3. $f(x) = \sqrt{x+3}$ at $(-2,1)$

$$f'(x) = \lim_{h\to 0} \left[\frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h}\right]\left[\frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}\right]$$

$$= \lim_{h\to 0} \frac{(x+h+3) - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \frac{1}{2\sqrt{x+3}}$$

$$f'(-2) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$\therefore (y-1) = \frac{1}{2}(x+2)$$

4. $f(x) = \dfrac{x+1}{2x}$ at $\left(2, \dfrac{3}{4}\right)$

$$f'(x) = \lim_{h\to 0} \frac{\frac{(x+h)+1}{2(x+h)} - \frac{x+1}{2x}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h+1)(x) - (x+1)(x+h)}{2(x+h)(x)(h)}$$

$$= \lim_{h\to 0} \frac{[\cancel{x^2} + \cancel{xh} + \cancel{x}] - [\cancel{x^2} + \cancel{x} + \cancel{xh} + h]}{2(x+h)(x)(h)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{2(x+h)(x)(\cancel{h})}$$

$$f'(x) = \frac{-1}{2x^2}$$

$$f'(2) = \frac{-1}{2(2)^2} = -\frac{1}{8}$$

Equation

$$\left(y - \frac{3}{4}\right) = \frac{-1}{8}(x-2)$$

#4

$$f(x) = \frac{1}{2x} + 3x$$

$$\lim_{h\to 0} \frac{\left(\frac{1}{2(x+h)} + 3(x+h)\right) - \left(\frac{1}{2x} + 3x\right)}{h} = \lim_{h\to 0} \frac{\frac{1+6(x+h)^2}{2(x+h)}_{(x)} - \frac{1+6x^2}{2x}_{(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{x + 6x(x+h)^2 - (1+6x^2)(x+h)}{2x(x+h).h} = \cancel{\;}$$

$$= \lim_{h\to 0} \frac{x + 6x(x^2+2xh+h^2) - (x+h+6x^3+6x^2h)}{2x(x+h).h}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + \cancel{6x^3} + 12x^2h + 6xh^2 - \cancel{x} - h - \cancel{6x^3} - 6x^2h}{2x(x+h).h}$$

$$= \lim_{h\to 0} \frac{6x^2h + 6xh^2 - h}{2x(x+h).h} = \lim_{h\to 0} \frac{6x^2 + 6xh - 1}{2x(x+h)}$$

$$= \frac{6x^2 - 1}{2x.x} = \frac{6x^2}{2x^2} - \frac{1}{2x^2} = \boxed{3 - \frac{1}{2x^2}} \Rightarrow f'(x) = 3 - \frac{1}{2x^2}$$

#3 $f(x) = \dfrac{1-2x}{3+x}$

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{1-2(x+h)}{3+(x+h)} - \dfrac{1-2x}{3+x}}{h}$$

$$= \lim_{h\to 0} \frac{(1-2x-2h)(3+x) - (1-2x)(3+x+h)}{(3+x+h)(3+x)(h)}$$

$$= \lim_{h\to 0} \frac{(\cancel{3}-\cancel{6x}-6h+\cancel{x}-\cancel{2x^2}-\cancel{2xh}) - (\cancel{3}+\cancel{x}+h-\cancel{6x}-\cancel{2x^2}-\cancel{2xh})}{(3+x+h)(3+x)(h)}$$

$$= \lim_{h\to 0} \frac{-7\cancel{h}}{(3+x)(3+x+h)\cancel{(h)}}$$

$$= \frac{-7}{(3+x)^2}$$