MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

WORKSHEET **MATH 1000** FALL 2013

SOLUTIONS

1. (a) First we find any vertical asymptotes. We set $(x-1)^3 = 0$ so $x = 1$. Note that the numerator is non-zero, so $x = 1$ is a vertical asymptote.

To find any horizontal asymptotes, we evaluate

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2(x+2)}{(x-1)^3} &= \lim_{x\to\infty} \frac{x^3+2x^2}{x^3-3x^2+3x-1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}} \\ &= \frac{1+0}{1-0+0-0} \\ &= 1. \end{aligned}$$

Thus $y = 1$ is a horizontal asymptote, and since $f(x)$ is a rational function, it must approach the horizontal asymptote both as $x \to \infty$ and as $x \to -\infty$.

To find any x-intercepts, we set $f(x) = 0$, so $x^2(x+2) = 0$, so $x = 0$ or $x = -2$. Thus the points $(0,0)$ and $(-2,0)$ are the x-intercepts. This also means that $(0,0)$ is the y-intercept, which we could alternatively find by evaluating $f(0)$.

Now we need to find any critical points. Note that $f'(x)$ is undefined only when $x = 1$ (the vertical asymptote) so we need only consider $f'(x) = 0$, that is, $-x(5x+4) = 0$, so $x = 0$ or $x = -\frac{4}{5}$. We can now construct the sign pattern found in Figure 1. We can see that $f(x)$ is increasing on the interval $\left(-\frac{4}{5}, 0\right)$ and decreasing on the intervals $\left(-\infty, -\frac{4}{5}\right)$, $(0,1)$ and $(1,\infty)$. Furthermore, we have a relative minimum at $x = -\frac{4}{5}$, which is the point $\left(-\frac{4}{5}, -\frac{32}{243}\right)$. We have a relative maximum at $x = 0$, which is the point $(0,0)$.

Figure 1: Sign patterns for Question 1(a).

Finally, we find the hypercritical points. Again, $f''(x)$ is undefined only at the vertical asymptote $x = 1$. Furthermore, $f''(x) = 0$ when $2(5x+1)(x+2) = 0$, that is when $x = -\frac{1}{5}$ or $x = -2$. We therefore construct the sign pattern found in Figure 1. We conclude that $f(x)$ is concave upward on the intervals $\left(-2, -\frac{1}{5}\right)$ and $(1,\infty)$ and concave downward on the intervals $(-\infty, -2)$ and $\left(-\frac{1}{5}, 1\right)$. The points of inflection occur at $x = -2$ and $x = -\frac{1}{5}$, which are the points $(-2,0)$ and $\left(-\frac{1}{5}, -\frac{1}{24}\right)$.

Now we can sketch the graph of $f(x)$, as found in Figure 2.

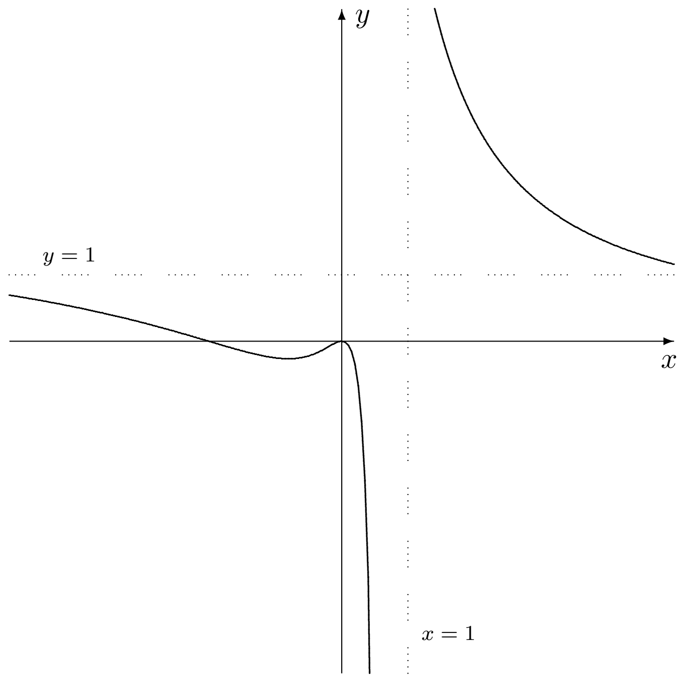

Figure 2: The graph for Question 1(a).

(b) To find the vertical asymptotes, we set $x^2+1=0$ so $x^2=-1$. This has no solutions, implying that there are no vertical asymptotes. For the horizontal asymptotes, we observe that $f(x)$ is rational so we need only take one limit at infinity:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2-4x+2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2-\frac{4}{x}+\frac{2}{x^2}}{1+\frac{1}{x^2}} = \frac{2-0+0}{1+0} = 2,$$

giving $y=2$ as the horizontal asymptote.

Now we seek the x-intercepts. Setting $f(x)=0$ gives

$$2x^2-4x+2 = 2(x-1)^2 = 0$$

so $x=1$; hence $(1,0)$ is the only x-intercept. Next we evaluate $f(0)=2$, so $(0,2)$ is the y-intercept.

Now we consider $f'(x)$. Observe that it never fails to exist since the denominator is always positive. Thus we set $f'(x)=0$ so

$$4x^2-4 = 4(x+1)(x-1) = 0$$

giving $x=1$ and $x=-1$ as the critical points. From the sign pattern given in Figure 3 we can see that $f(x)$ is increasing on $(-\infty,-1)\cup(1,\infty)$, while it is decreasing on $(-1,1)$. Hence the point $(-1,4)$ is a relative maximum while the point $(1,0)$ is a relative minimum.

Finally, we consider concavity. Again, $f''(x)$ never fails to exist so we set $f''(x)=0$, giving

$$24x-8x^3 = 8x(x^2-3) = 0.$$

Figure 3: The sign patterns for Question 1(b).

Thus $x = 0$, $x = \sqrt{3}$ and $x = -\sqrt{3}$ are the hypercritical points. From the sign pattern, we see that $f(x)$ is concave upward on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$, and it is concave downward on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$. Hence the points $(-\sqrt{3}, 2 + \sqrt{3})$, $(0, 2)$ and $(\sqrt{3}, 2 - \sqrt{3})$ are all inflection points.

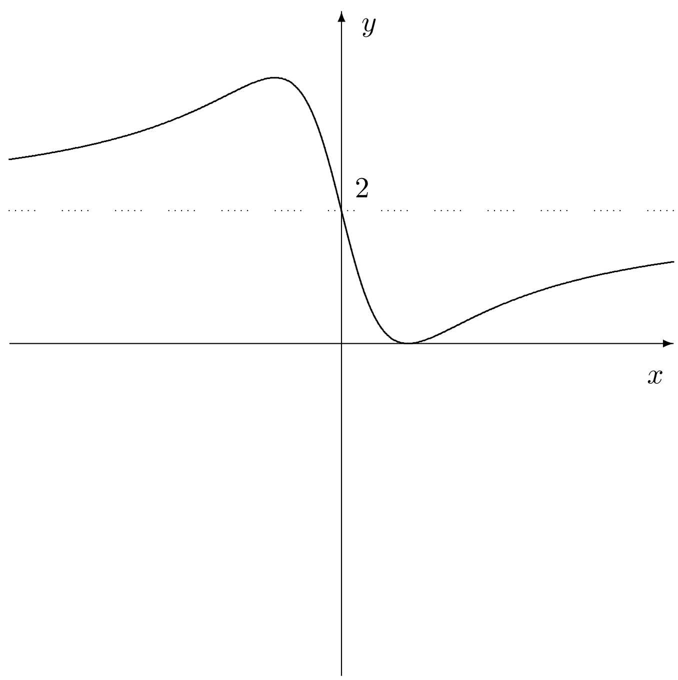

Figure 4: The graph for Question 1(b).

We can now sketch the graph depicted in Figure 4.

2. (a) First we determine the critical points of $f(x)$. We have

$$f'(x) = 3x^2 - 9 = 3(x^2 - 3)$$

so $f'(x) = 0$ when $x = \pm\sqrt{3}$ and $f'(x)$ is never undefined. Hence $x = \pm\sqrt{3}$ are the only critical points. Next we evaluate $f(x)$ at the critical points and at the endpoints $x = -4$

and $x = 3$:

$$f\left(-\sqrt{3}\right) = 6\sqrt{3} \approx 10.4, \quad f\left(\sqrt{3}\right) = -6\sqrt{3} \approx -10.4$$
$$f(-4) = -28, \quad f(3) = 0.$$

Hence the maximum value of $f(x)$ on $[-4, 3]$ is $6\sqrt{3}$ and the minimum value is -28.

(b) First we identify the critical points. Note that

$$f'(x) = \frac{(2x)(x+1) - (1)(x^2+3)}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2},$$

which is zero for $x = 1$ and $x = -3$, and fails to exist at $x = -1$. For the interval $[0, 4]$, then, the only critical point is $x = 1$, for which $f(1) = 2$. Checking the endpoints, we have $f(0) = 3$ and $f(4) = \frac{19}{5} = 3.8$. Hence the maximum value of $f(x)$ is $\frac{19}{5}$, and the minimum value is 2.

(c) As before, we find the critical points. Differentiation gives

$$f'(x) = 1 + 2\sin(x)$$

which fails to exist nowhere, and is zero for $x = -\frac{\pi}{6}$ and $x = -\frac{5\pi}{6}$, both of which are on the interval $[-\pi, \pi]$. Note that

$$f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\cos\left(-\frac{\pi}{6}\right) \approx -2.26$$

and

$$f\left(-\frac{5\pi}{6}\right) = -\frac{5\pi}{6} - 2\cos\left(-\frac{5\pi}{6}\right) \approx -0.89.$$

At the endpoints,

$$f(-\pi) = -\pi - 2\cos(-\pi) \approx -1.14 \quad \text{and} \quad f(\pi) = \pi - 2\cos(\pi) \approx 5.14.$$

Hence the maximum value of $f(x)$ on $[-\pi, \pi]$ is approximately 5.14, while the minimum value is about -2.26.

(d) We again begin by determining the critical points of $f(x)$, observing that

$$f'(x) = \sec(x)\tan(x).$$

Setting $f'(x) = 0$ gives $x = 0$, and $f'(x)$ never fails to exist on the given interval (though it does fail to exist for many other values of x). Hence $x = 0$ is the only critical point. We evaluate $f(x)$ there and at the endpoints, giving

$$f(0) = 1, \quad f\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3} \approx 1.15, \quad f\left(\frac{\pi}{3}\right) = 2.$$

Thus the maximum value of $f(x)$ on the given interval is 2 and the minimum value is 1.

3. Let x be the length of fencing parallel to the river and y be the length of the other side of the rectangle. The quantity to be maximised is the area A. The primary equation is $A = xy$ and the secondary equation is

$$x + 2y = 1000 \quad \Longrightarrow \quad x = 1000 - 2y$$

so the reduced primary equation is

$$A(y) = (1000 - 2y)y = 1000y - 2y^2.$$

This problem is defined on an open interval, so we next compute

$$A'(y) = 1000 - 4y$$

and set $A'(y) = 0$, giving $y = 250$. Note that $A''(y) = -4$ so $A''(y) < 0$ for all y, and in particular for $y = 250$. Hence, by the Second Derivative Test, $y = 250$ is the absolute maximum. When $y = 250$, from the secondary equation we see that

$$x = 1000 - 2(250) = 500.$$

Thus the area is a maximum when the plot of land measures 500 metres by 250 metres.

4. Let r be the radius of the cylinder (and thus also of the hemisphere) and h be its height. The quantity to be minimised is the cost C. Note that the surface area of the cylindrical portion (including the bottom) is $\pi r^2 + 2\pi rh$ (the normal surface area of a cylinder, minus the surface area of the circle at the top) while the surface area of the hemisphere is $2\pi r^2$ (half the surface area of a sphere). Hence the primary equation is

$$C = 2(\pi r^2 + 2\pi rh) + 3.5(2\pi r^2) = 9\pi r^2 + 4\pi rh.$$

Since the volume of a cylinder is $\pi r^2 h$ and the volume of a hemisphere is $\frac{2\pi}{3}r^3$ (again, half the volume of a sphere), the secondary equation is

$$\pi r^2 h + \frac{2\pi}{3}r^3 = 1 \quad \Longrightarrow \quad h = \frac{1 - \frac{2\pi}{3}r^3}{\pi r^2}.$$

Thus the reduced primary equation is

$$C(r) = 9\pi r^2 + 4\pi r\left(\frac{1 - \frac{2\pi}{3}r^3}{\pi r^2}\right) = 9\pi r^2 + \frac{4}{r} - \frac{8\pi}{3}r^2 = \frac{19\pi}{3}r^2 + \frac{4}{r}.$$

Observe that

$$C'(r) = \frac{38\pi}{3}r - \frac{4}{r^2},$$

and setting $C'(r) = 0$ yields

$$\frac{38\pi}{3}r = \frac{4}{r^2} \quad \Longrightarrow \quad r^3 = \frac{6}{19\pi} \quad \Longrightarrow \quad r = \sqrt[3]{\frac{6}{19\pi}}.$$

Also,

$$C''(r) = \frac{38\pi}{3} + \frac{8}{r^3} \quad \Longrightarrow \quad C''\left(\sqrt[3]{\frac{6}{19\pi}}\right) > 0,$$

so this value of r is the absolute minimum. The minimum value of C, then, is

$$C\left(\sqrt[3]{\frac{6}{19\pi}}\right) = \frac{19\pi}{3}\left(\sqrt[3]{\frac{6}{19\pi}}\right)^2 + \frac{4}{\sqrt[3]{\frac{6}{19\pi}}} \approx 12.90,$$

which means that the cheapest possible cost of the tube is \$12.90.

5. Let x and y be the length and height of the poster; see Figure 5.

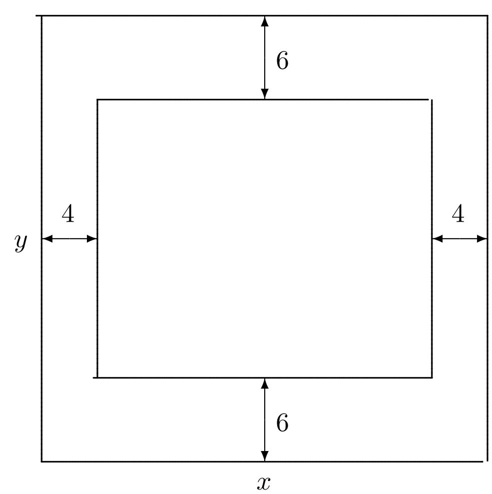

Figure 5: A poster with margins, as discussed in Question 3.

The quantity to be minimised is its area, A. The primary equation is

$$A = xy.$$

Note that the length of the printed matter, accounting for the side margins, is $x - 8$ while the height is $y - 12$. Hence the secondary equation is

$$(x-8)(y-12) = 384 \quad \Longrightarrow \quad y = \frac{384}{x-8} + 12.$$

The reduced primary equation is

$$A(x) = x\left(\frac{384}{x-8} + 12\right) = \frac{384x}{x-8} + 12x \quad \Longrightarrow \quad A' = \frac{12(x^2 - 16x - 192)}{(x-8)^2}.$$

We set $A' = 0$ and get $x = 24$ and $x = -8$, but we can disregard the latter because length must be positive. Note that

$$A'' = \frac{6144}{(x-8)^3} \quad \Longrightarrow \quad A''(24) > 0$$

so by the Second Derivative Test, $x = 24$ is an absolute minimum. By the secondary equation, when $x = 24$,

$$y = \frac{384}{24-8} + 12 = 36,$$

so the dimensions of the poster with the smallest area are 24 cm $\times$ 36 cm.

6. Let the distance between Toontown and the Roadrunner be r, and the distance between Wile E Coyote and Toontown be c. The quantity to be minimised is the distance between the Roadrunner and Wile E Coyote, ℓ, as shown in Figure 6. Note that, because he arrives in Toontown at 3:00pm after travelling at 15 km/hr, at 2:00pm Wile E Coyote must be 15 km west of Toontown.

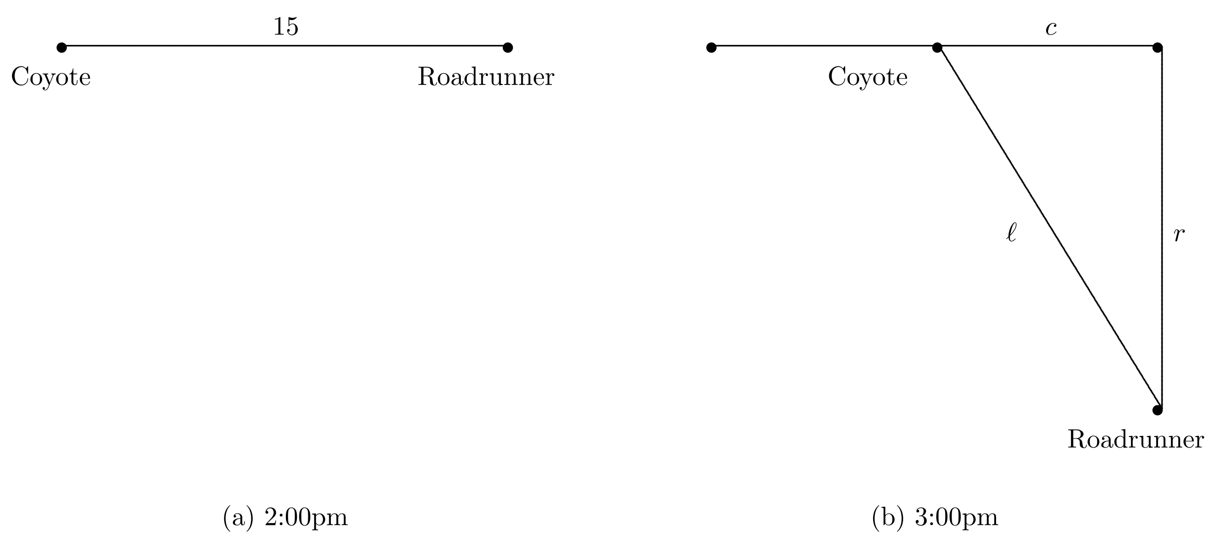

Figure 6: Wile E Coyote fails to catch the Roadrunner, as in Question 4.

The primary equation is

$$\ell = \sqrt{c^2 + r^2}.$$

To find secondary equations, let t be the time (measured in hours) elapsed since 2:00pm. Then

$$r = 20t$$

and

$$c = 15 - 15t.$$

Hence the primary equation becomes

$$\ell(t) = \sqrt{(15-15t)^2 + (20t)^2} = \sqrt{625t^2 - 450t + 225}.$$

This is defined on the closed interval $[0,1]$, from when Wile E Coyote launches himself towards Toon Town, to when he arrives. Observe that

$$\ell'(t) = \frac{5(50t-18)}{2\sqrt{25t^2-18t+9}}.$$

If we set this equal to zero, we obtain

$$50t - 18 = 0 \quad \Longrightarrow \quad t = \frac{9}{25}.$$

Note that

$$\ell\left(\frac{9}{25}\right) = 12,$$

while at the endpoints,

$$\ell(0) = 15 \quad \text{and} \quad \ell(1) = 20.$$

Hence the distance between the Roadrunner and Wile E Coyote will be the smallest after $\frac{9}{25}$ hours.

7. (a) This is a $\frac{0}{0}$ indeterminate form:

$$\lim_{x\to 0} \frac{6^x - 2^x}{x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{6^x \ln(6) - 2^x \ln(2)}{1} = \ln(6) - \ln(2) = \ln(3).$$

(b) This is a $\frac{0}{0}$ indeterminate form:

$$\lim_{x\to 0^+} \frac{1-\cos\left(\sqrt{x}\right)}{x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\sin\left(\sqrt{x}\right)\cdot \frac{1}{2\sqrt{x}}}{1} = \lim_{x\to 0^+} \frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

$$\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\cos\left(\sqrt{x}\right)\cdot \frac{1}{2\sqrt{x}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\cos\left(\sqrt{x}\right)}{2} = \frac{1}{2}.$$

(c) This is a $\frac{0}{0}$ indeterminate form:

$$\lim_{x\to 0} \frac{\sin(mx)}{\sin(nx)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{m\cos(mx)}{n\cos(nx)} = \frac{m\cdot 1}{n\cdot 1} = \frac{m}{n}.$$

(d) This is an $\frac{\infty}{\infty}$ indeterminate form:

$$\lim_{x\to\infty} \frac{\ln(1+e^{2x})}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+e^{2x}}\cdot 2e^{2x}}{1} = \lim_{x\to\infty} \frac{2e^{2x}}{1+e^{2x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{4e^{2x}}{2e^{2x}} = \lim_{x\to\infty} 2 = 2.$$

(e) This is an $\frac{\infty}{\infty}$ indeterminate form:

$$\begin{aligned}
\lim_{x\to\infty} \frac{[\ln(x)]^3}{x^2} &\stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{3[\ln(x)]^2 \cdot \frac{1}{x}}{2x} \\
&= \lim_{x\to\infty} \frac{3[\ln(x)]^2}{2x^2} \\
&\stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{6\ln(x) \cdot \frac{1}{x}}{4x} \\
&= \lim_{x\to\infty} \frac{3\ln(x)}{2x^2} \\
&\stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{3 \cdot \frac{1}{x}}{4x} \\
&= \lim_{x\to\infty} \frac{3}{4x^2} \\
&= 0.
\end{aligned}$$

(f) This is an $\frac{\infty}{\infty}$ indeterminate form:

$$\lim_{x\to\infty} \frac{x^2+1}{x\ln(x)} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{2x}{x\left(\frac{1}{x}\right)+\ln(x)} = \lim_{x\to\infty} \frac{2x}{1+\ln(x)} \stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{2}{\frac{1}{x}} = \lim_{x\to\infty} 2x = \infty.$$

(g) This is an $\infty \cdot 0$ indeterminate form:

$$\lim_{x\to\frac{\pi}{2}^-} \sec(7x)\cos(3x) = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos(3x)}{\cos(7x)} \stackrel{\text{H}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-3\sin(3x)}{-7\sin(7x)} = \frac{-3(-1)}{-7(-1)} = \frac{3}{7}.$$

(h) This is an $\infty - \infty$ indeterminate form:

$$\begin{aligned}
\lim_{x\to 1} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) &= \lim_{x\to 1} \frac{x-1-\ln(x)}{(x-1)\ln(x)} \\
&\stackrel{\text{H}}{=} \lim_{x\to 1} \frac{1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x}+\ln(x)} \\
&= \lim_{x\to 1} \frac{x-1}{x-1+x\ln(x)} \\
&\stackrel{\text{H}}{=} \lim_{x\to 1} \frac{1}{1+\ln(x)+x\cdot\frac{1}{x}} \\
&= \lim_{x\to 1} \frac{1}{2+\ln(x)} \\
&= \frac{1}{2}.
\end{aligned}$$

(i) This is a 0^0 indeterminate form. Let $y = \sin(x)^{\tan(x)}$ so $\ln(y) = \tan(x)\ln(\sin(x))$. Then

$$\begin{aligned}\lim_{x\to 0^+} \tan(x)\ln(\sin(x)) &= \lim_{x\to 0^+} \frac{\ln(\sin(x))}{\cot(x)} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin(x)}[\cos(x)]}{-\csc^2(x)} \\ &= \lim_{x\to 0^+} [-\sin(x)\cos(x)] = 0.\end{aligned}$$

Thus $\lim_{x\to 0^+} (\sin(x))^{\tan(x)} = e^0 = 1$.

(j) This is an ∞^0 indeterminate form. Let $y = (x+e^x)^{\frac{1}{x}}$ so $\ln(y) = \frac{1}{x}\cdot\ln(x+e^x)$. Then

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x}\cdot\ln(x+e^x) &= \lim_{x\to\infty} \frac{\ln(x+e^x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+e^x}\cdot(1+e^x)}{1} = \lim_{x\to\infty} \frac{1+e^x}{x+e^x} \\ &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{1+e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1.\end{aligned}$$

Thus $\lim_{x\to\infty} (x+e^x)^{\frac{1}{x}} = e^1 = e$.

(k) This is a 1^∞ indeterminate form. Let $y = \cos(3x))^{\frac{5}{x}}$ so $\ln(y) = \frac{5}{x}\ln(\cos(3x))$. Then

$$\begin{aligned}\lim_{x\to 0} \frac{5}{x}\ln(\cos(3x)) &= \lim_{x\to 0} \frac{5\ln(\cos(3x))}{x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{5\cdot\frac{1}{\cos(3x)}\cdot[-3\sin(3x)]}{1} \\ &= \lim_{x\to 0}[-15\tan(3x)] = 0.\end{aligned}$$

Thus $\lim_{x\to 0}(\cos(3x))^{\frac{5}{x}} = e^0 = 1$.

(ℓ) This is a 1^∞ indeterminate form. Let $y = \left(1+\frac{a}{x}\right)^{bx}$ so $\ln(y) = bx\ln\left(1+\frac{a}{x}\right)$. Then

$$\lim_{x\to\infty} bx\ln\left(1+\frac{a}{x}\right) = \lim_{x\to\infty} \frac{b\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{b}{1+\frac{a}{x}}\cdot\left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{ab}{1+\frac{a}{x}} = ab.$$

Thus $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = e^{ab}$.