MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION | **Mathematics 1000** | FALL 1993

Marks

[9] 1. Evaluate each limit if it exists. Assign $+\infty$ or $-\infty$, if possible, to any limit that does not exist.

(a) $\lim_{x\to 2} \dfrac{2x^2-3x-2}{x^4-8x}$ (b) $\lim_{x\to 7} \dfrac{3-\sqrt{x+2}}{2x-14}$ (c) $\lim_{x\to 4^+} \dfrac{x^2+3}{x^2-9x+20}$

[7] 2. Let $f(x) = \begin{cases} Ax-3 & , \; x \le 1 \\ 5A/(x-2) & , \; x > 1 \end{cases}$

(a) Find the constant A such that f is continuous at $x = 1$.

(b) For which nonzero values (if any) of A is f continuous at $x = 2$? Explain your answer.

[6] 3. Use the definition of derivative to find $f'(x)$ if $f(x) = \dfrac{3}{x}$.

[20] 4. For each of the following, find y' and make any obvious simplifications.

(a) $y = \dfrac{x^6}{(1-2x^2)^3}$ (b) $y = x^2 \sin^2 x$

(c) $y = \sqrt{1 - 2\cos(1-2x)}$ (d) $y = \ln(1 - e^{-x^2})$

[7] 5. Find all values of the constant a such that the curve $xy^2 + 4 = a(2x+y)$ has a horizontal tangent at the point $P(-1,2)$.

[8] 6. A trough is 8 metres long and its ends have the shape of isosceles triangles that are 3 metres across the top and have a height of 2 metres. If water is poured into the trough at a rate of 8 m^3/min, how fast does the water level rise when the water is 50 cm. deep?

[11] 7. Sketch the graph of $y = 3x^5 - 5x^3$, labelling intercepts, extrema and points of inflection.

[9] 8. Find the area of the largest rectangle that has its base on the x-axis and its other two vertices above the x-axis and on the parabola $y = 27 - x^2$.

[9] 9. Evaluate each of the following integrals:

(a) $\int (6\sqrt{x} - \frac{1}{3x})dx$

(b) $\int (\sin(2x+1) - 3e^{4x})dx$

(c) $\int_0^{\frac{\pi}{4}} \sec^2 x dx$

[7] 10. Find the area of the region bounded by the curves $y = -x^2 - 2x$ and $y = x + 2$.

[7] 11. Attempt exactly ONE of the following:

(a) Using the definition of derivative, prove the Product Rule.

(b) Find the point on the parabola $y = x^2$ that is closest to the point $(16, \frac{1}{2})$.

(c) Prove that if an object moves in a straight line with a constant acceleration, then it reverses its direction at most once.

[100]

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EXAM SOLUTIONS — Mathematics 1000 — FALL 1993

1. (a) $\lim_{x\to 2}\frac{2x^2-3x-2}{x^4-8x} = \lim_{x\to 2}\frac{(x-2)(2x+1)}{x(x-2)(x^2+2x+4)} = \lim_{x\to 2}\frac{2x+1}{x(x^2+2x+4)} = \frac{5}{24}.$

   (b)
$$\begin{aligned}\lim_{x\to 7}\frac{3-\sqrt{x+2}}{2x-14} &= \lim_{x\to 7}\frac{(3-\sqrt{x+2})(3+\sqrt{x+2})}{2(x-7)(3+\sqrt{x+2})}\\ &= \lim_{x\to 7}\frac{9-(x+2)}{2(x-7)(3+\sqrt{x+2})} = \lim_{x\to 7}\frac{7-x}{2(x-7)(3+\sqrt{x+2})}\\ &= \lim_{x\to 7}\frac{-1}{2(3+\sqrt{x+2})} = \frac{-1}{2(3+\sqrt{9})} = -\frac{1}{12}\end{aligned}$$

   (c) $\lim_{x\to 4^+}\frac{x^2+3}{x^2-9x+20} = \lim_{x\to 4^+}\frac{x^2+3}{(x-4)(x-5)}\left[\frac{+}{(+)(-)}\right] = -\infty$

2. (a) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(Ax-3) = A-3$

   $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\frac{5A}{x-2} = \frac{5A}{-1} = -5A$

   For $\lim_{x\to 1} f(x)$ to exist, we must have
$$\begin{aligned}A-3 &= -5A\\ A &= \frac{1}{2}\end{aligned}$$

   Then $\lim_{x\to 1^-} f(x) = \frac{1}{2}-3 = -\frac{5}{2}$ and $\lim_{x\to 1^+} f(x) = -5(\frac{1}{2}) = -\frac{5}{2}$ and so $\lim_{x\to 1} f(x) = -\frac{5}{2}$. Also, $f(1) = A(1)-3 = \frac{1}{2}(1)-3 = -\frac{5}{2}$. Thus, $\lim_{x\to 1} f(x) = f(1)$ and f is continuous at $x=1$.

   (b) $f(2)$ does not exist since $f(x) = \frac{5A}{x-2}$ for $x>1$. So f is not continuous at $x=2$.

3.
$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{3}{x+h}-\frac{3}{x}}{h} = \lim_{h\to 0}\frac{\frac{3x}{x(x+h)}-\frac{3(x+h)}{x(x+h)}}{h} = \lim_{h\to 0}\frac{3x-3(x+h)}{x(x+h)h}\\ &= \lim_{h\to 0}\frac{3x-3x-3h}{x(x+h)h} = \lim_{h\to 0}\frac{-3h}{x(x+h)h} = \lim_{h\to 0}\frac{-3}{x(x+h)} = -\frac{3}{x^2}\end{aligned}$$

4. (a)
$$\begin{aligned}y &= \frac{x^6}{(1-2x^2)^3}\\ y' &= \frac{(1-2x^2)^3(6x^5)-x^6[3(1-2x^2)^2(-4x)]}{((1-2x^2)^3)^2} = \frac{(1-2x^2)^3(6x^5)+12x^7(1-2x^2)^2}{(1-2x^2)^6}\\ &= \frac{6x^5(1-2x^2)^2(1-2x^2+2x^2)}{(1-2x^2)^6} = \frac{6x^5}{(1-2x^2)^4}\end{aligned}$$

(b) $y = x^2 \sin^2 x$

$y' = x^2(2 \sin x \cos x) + 2x \sin^2 x = 2x \sin x(x \cos x + \sin x)$

(c) $y = \sqrt{1 - 2\cos(1 - 2x)}$

$y' = \frac{1}{2}[1 - 2\cos(1-2x)]^{-\frac{1}{2}}[2\sin(1-2x)](-2) = -\frac{2\sin(1-2x)}{\sqrt{1-2\cos(1-2x)}}$

(d) $y = \ln(1 - e^{-x^2})$

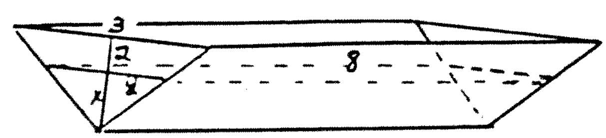
$y' = \left(\frac{1}{1-e^{-x^2}}\right)\left(-e^{-x^2}\right)(-2x) = \frac{2xe^{-x^2}}{1-e^{-x^2}}$

5. $x(2yy') + y^2 = a(2 + y')$
$2xyy' - ay' = 2a - y^2$
$y'(2xy - a) = 2a - y^2$
$y' = \frac{2a - y^2}{2xy - a}$

For a horizontal tangent at $(-1, 2)$, we must have $y' = 0$ when $x = -1, y = 2$.

$$\begin{aligned} \frac{2a - 2^2}{2(-1)(2) - a} &= 0 \\ 2a - 4 &= 0 \\ a &= 2 \end{aligned}$$

6. Let x = height of water and y = length across top of water as shown.
$\therefore \frac{y}{x} = \frac{3}{2}$, so $y = \frac{3x}{2}$. Then the volume of water is



$V = \frac{1}{2}y(x)(8) = 4xy = 4x(\frac{3x}{2}) = 6x^2.$
$V' = 12xx'.$

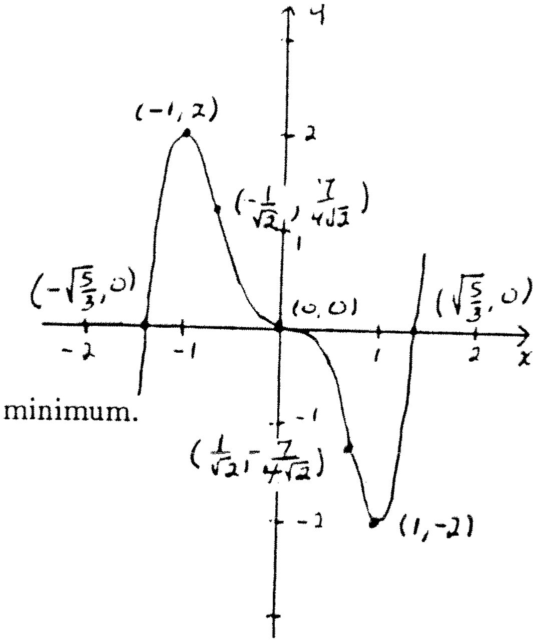
We are given $V' = 8$ m^3/min and want to find x' when $x = 50$ cm $= \frac{1}{2}$ m.

$\therefore x' = \frac{V'}{12x} = \frac{8}{12(\frac{1}{2})} = \frac{4}{3}$ m/min.

7. $y = 3x^5 - 5x^3$

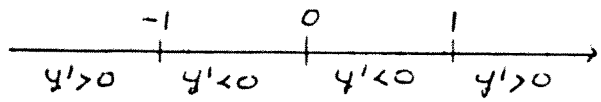
Intercepts:

$x = 0 \Rightarrow y = 0$
$y = 0 \Rightarrow 3x^5 - 5x^3 = 0$
$x^3(3x^2 - 5) = 0$
$x = 0, x = \pm\sqrt{5/3}$



Local Extrema:

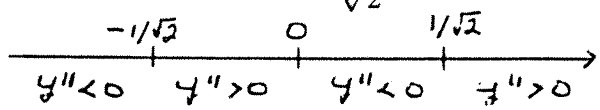
$$y' = 15x^4 - 15x^2 = 15x^2(x^2-1)$$
$$y' = 0 \Rightarrow x = 0, x = \pm 1$$



So $(-1,2)$ is a local maximum and $(1,-2)$ is a local minimum.

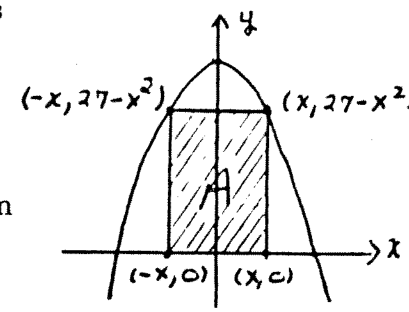
Inflection Points:

$$y'' = 60x^3 - 30x = 30x(2x^2-1)$$
$$y'' = 0 \Rightarrow x = 0, x = \pm\frac{1}{\sqrt{2}}$$



So $(-\frac{1}{\sqrt{2}}, \frac{7}{4\sqrt{2}})$, $(0,0)$ and $(\frac{1}{\sqrt{2}}, -\frac{7}{4\sqrt{2}})$ are inflection points.

8. If we let the lower two vertices of the rectangle be $(x,0)$ and $(-x,0)$, then the upper two vertices are $(x, 27-x^2)$ and $(-x, 27-x^2)$ as shown. The area is



$$A(x) = 2x(27-x^2) = 54x - 2x^3$$
$$A'(x) = 54 - 6x^2 = 6(9-x^2)$$
$$A'(x) = 0 \text{ when } x = 3 (x > 0)$$

Now $A''(x) = -12x \Rightarrow A''(3) = -36 < 0$, so the area is maximum at $x = 3$, and the area is $A(3) = 54(3) - 2(3)^3 = 108$.

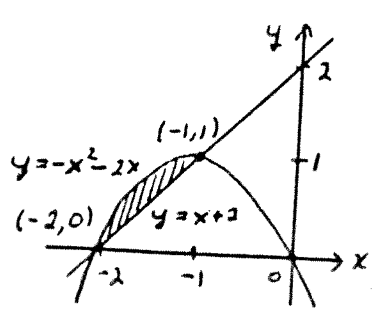
9. (a) $\int \left(6\sqrt{x} - \frac{1}{3x}\right) dx = \int (6x^{\frac{1}{2}} - \frac{1}{3}x^{-1})dx = 6\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{3}\ln x + C = 4x^{\frac{3}{2}} - \frac{1}{3}\ln x + C$

(b) $\int (\sin(2x+1) - 3e^{4x})dx = -\frac{1}{2}\cos(2x+1) - \frac{3}{4}e^{4x} + C$

(c) $\int_0^{\frac{\pi}{4}} \sec^2 x dx = [\tan x]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

10. Points of Intersection:

$y = -x^2 - 2x, y = x + 2$
$-x^2 - 2x = x + 2$
$x^2 + 3x + 2 = 0$
$(x+1)(x+2) = 0$
$x = -1, x = -2$
$(-1,1), (-2,0)$



$$\begin{aligned} A &= \int_{-2}^{-1} [(-x^2 - 2x) - (x+2)]dx = \int_{-2}^{1} (-x^2 - 3x - 2)dx = \left(-\frac{x^3}{3} - \frac{3x^2}{2} - 2x\right)\Big|_{-2}^{-1} \\ &= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{8}{3} - 6 + 4\right) = \frac{5}{6} - \frac{2}{3} = \frac{1}{6} \end{aligned}$$

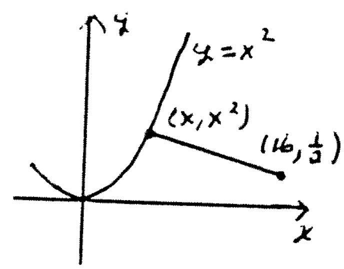
11. (a) Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \\ &= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right] \\ &= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

(b) A typical point on the parabola is (x, x^2).

The distance from (x, x^2) to $\left(16, \frac{1}{2}\right)$ is $\sqrt{(x-16)^2 + (x^2 - \frac{1}{2})^2}$.

We can minimize this quantity by minimizing its square



$$\begin{aligned} f(x) &= (x-16)^2 + (x^2 - \frac{1}{2})^2 \\ f'(x) &= 2(x-16) + 2(x^2 - \frac{1}{2})(2x) \\ &= 4x^3 - 2x + 2x - 32 \\ &= 4x^3 - 32 \\ f'(x) &= 0 \text{ when } x^3 = 8, \text{ so } x = 2. \end{aligned}$$

Now $f''(x) = 12x^2 \Rightarrow f''(2) = 48$. So f has a minimum at $x = 2$ and the required point is $(2, 4)$.

(c) Let the acceleration be A. Then the velocity is given by

$$v(t) = \int A dt = At + C$$

where C is constant.

An object reverses direction when $v(t) = 0$. But $v(t) = 0 \Rightarrow At + C = 0 \Rightarrow t = -\frac{C}{A}$.
Since the velocity is zero for only one value of t, the object reverses direction at most once.