MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION
TIME: TWO HOURS

Mathematics 1000

WINTER 1993

ANSWER ALL QUESTIONS. SHOW ALL WORK

Marks

[9] 1. Evaluate each limit.

(a) $\lim\limits_{x \to -2} \dfrac{x^2 + x - 2}{20 - 5x^2}$

(b) $\lim\limits_{x \to 0} \dfrac{2 - \sqrt{4 - x}}{x}$

(c) $\lim\limits_{x \to -\infty} \dfrac{3x + 4}{\sqrt{2x^2 - 5}}$

[5] 2. Find the equation of all vertical and horizontal asymptotes to the graph of

$$f(x) = \frac{x^2 + 2x - 3}{x^2 - 3x + 2}.$$

Give a brief justification in terms of limits but do not sketch the graph of $f(x)$.

[10] 3. (a) Define what it means for the function $f(x)$ to be continuous at $x = c$.

(b) Using the definition, find the value of k for which $f(x)$ is continuous at $x = 1$ where

$$f(x) = \begin{cases} x^3 - x + 1 & \text{if } x \le 1, \\ k\sqrt{x + 3} & \text{if } x > 1. \end{cases}$$

[6] 4. Using the definition of derivative, find $f'(x)$ if $f(x) = 1 - \dfrac{1}{2 - x}$.

[16] 5. For each of the following functions, find $f'(x)$ and make any obvious simplifications.

(a) $f(x) = \dfrac{x^3 + 2}{x^2 + 2}$

(b) $f(x) = e^{-\frac{x}{2}} \tan(x^2)$

(c) $f(x) = \sqrt{1 + \sin(2x)}$

(d) $f(x) = \dfrac{1 + \ln x}{1 + x}$

[6] 6. Find the equation of the normal line to the curve $x^2y + y^3 = 2x + 4$ at the point $(-1, 1)$.

[10] 7. A boat sails parallel to a straight beach at a constant rate of 12 mi/hr, staying 4 miles offshore. At what rate is the distance between the boat and a lighthouse on the shoreline changing at the instant the boat is 5 miles from the lighthouse?

[12] 8. Sketch the graph of $f(x) = 3x^4 - 4x^3$. Label all intercepts, local extrema, and points of inflection. Show all work.

[10] 9. If a rectangular box with a square base and an open top is to have a volume of 4 cubic metres, find the dimensions that require the least amount of material. (Neglect the thickness of the material and waste in construction.)

[16] 10. (a) Evaluate each integral:

(i) $\int \left(\frac{1}{\sqrt[3]{x^2}} - 3x^{\frac{3}{2}} + \frac{1}{2x} \right) dx$

(ii) $\int (3\cos 4x - 4e^{3x} - \sin 2x) dx$

(iii) $\int_1^5 \frac{1}{3x-2} dx$

(b) Find the area of the region bounded by the curves $y = 2 - x^2$ and $y = -x$.

[100]

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1. (a) $\lim\limits_{x\to -2} \dfrac{x^2+x-2}{20-5x^2} = \lim\limits_{x\to -2} \dfrac{(x+2)(x-1)}{5(2-x)(2+x)} = \lim\limits_{x\to -2} \dfrac{x-1}{5(2-x)} = -\dfrac{3}{20}$

   (b) $\lim\limits_{x\to 0} \dfrac{2-\sqrt{4-x}}{x} \cdot \dfrac{2+\sqrt{4-x}}{2+\sqrt{4-x}} = \lim\limits_{x\to 0} \dfrac{4-(4-x)}{x(2+\sqrt{4-x})} = \lim\limits_{x\to 0} \dfrac{x}{x(2+\sqrt{4-x})} = \dfrac{1}{4}$

   (c) $\lim\limits_{x\to -\infty} \dfrac{3x+4}{\sqrt{2x^2-5}} = \lim\limits_{x\to -\infty} \dfrac{x(3+\frac{4}{x})}{\sqrt{x^2(2-\frac{5}{x^2})}} = \lim\limits_{x\to -\infty} \dfrac{x(3+\frac{4}{x})}{-x\sqrt{2-\frac{5}{x^2}}} = -\dfrac{3}{\sqrt{2}}$

2. $f(x) = \dfrac{(x+3)(x-1)}{(x-2)(x-1)} = \dfrac{x+3}{x-2} \qquad x \neq 1$

   $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x+3}{x-2} = 1 \qquad \therefore y = 1$ is a Horizontal Asymptote

   $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \dfrac{x+3}{x-2} = \infty \qquad \therefore x = 2$ is a Vertical Asymptote

x	y
2.1	$\dfrac{5.1}{.1} = 51$

   Note: $x = 1$ is a removable discontinuity

3. (a) If $\lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^-} f(x) = f(c)$, then $f(x)$ is continuous at $x = c$.

   (b) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^3 - x + 1) = 1 = f(1)$

   $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} k\sqrt{x+3} = 2k$

   Must have $2k = 1$ or $k = \dfrac{1}{2}$ for continuity.

4. $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[1-\frac{1}{2-(x+h)}]-[1-\frac{1}{2-x}]}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2-x}-\frac{1}{2-(x+h)}}{h} = \lim_{h\to 0} \frac{2-x-h-2+x}{(2-x)[2-(x+h)]h} = \lim_{h\to 0} \frac{-h}{(2-x)[2-(x+h)]h} \\ &= \lim_{h\to 0} \frac{-1}{(2-x)[2-(x+h)]} = -\frac{1}{(2-x)^2} \end{aligned}$$

5. (a) $f(x) = \dfrac{x^3+2}{x^2+2}$

   $f'(x) = \dfrac{(x^2+2)(3x^2)-(x^3+2)(2x)}{(x^2+2)^2} = \dfrac{x[3x(x^2+2)-2(x^3+2)]}{(x_2^2)^2} = \dfrac{x(x^3+6x-4)}{(x^2+2)^2}$

(b) $f(x) = e^{-\frac{x}{2}}\tan(x^2)$

$$f'(x) = e^{-\frac{x}{2}}\sec^2(x^2)2x + \tan(x^2)e^{-\frac{x}{2}}(-\frac{1}{2}) = \frac{1}{2}e^{-\frac{x}{2}}[4x\sec^2(x^2) - \tan(x^2)]$$

(c) $f(x) = \sqrt{1+\sin(2x)}$

$$f'(x) = \frac{1}{2}(1+\sin(2x))^{-\frac{1}{2}}\cos(2x)2 = \frac{\cos(2x)}{\sqrt{1+\sin(2x)}}$$

(d) $f(x) = \dfrac{1+\ln x}{1+x}$

$$f'(x) = \frac{(1+x)\frac{1}{x} - (1+\ln x)}{(1+x)^2} = \frac{(1+x) - x(1+\ln x)}{x(1+x)^2} = \frac{1 - x\ln x}{x(1+x)^2}$$

6. Using implicit differentiation:

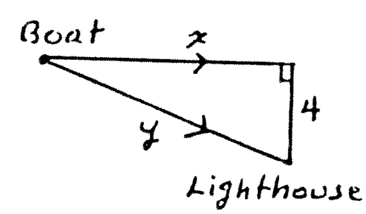
$$\begin{aligned} x^2y + y^3 &= 2x+4 \\ x^2y' + 2xy + 3y^2y' &= 2 \\ (x^2+3y^2)y' &= 2-2xy \\ y' &= \frac{2-2xy}{x^2+3y^2} \end{aligned}$$

At $(-1,1)$: $y' = \dfrac{2-2(-1)(1)}{(-1)^2+3(1)^2} = \dfrac{4}{4} = 1.$

So, the slope of the normal line is $m_N = -1$, and its equation is

$$\begin{aligned} y-1 &= -1(x+1) \\ y &= -x \end{aligned}$$

7. $\dfrac{dx}{dt} = -12$, find $\dfrac{dy}{dt}$ when $y = 5$.



$y^2 = x^2 + 16$

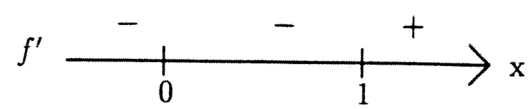
$2yy' = 2xx'$

$y' = \dfrac{xx'}{y}$

When $y = 5, x = \sqrt{25-16} = \sqrt{9} = 3$, and then $y' = \dfrac{3(-12)}{5} = -\dfrac{36}{5}$

Distance is decreasing at $\frac{36}{5}$ mi/hr. (7.2 mi/hr.)

8. $f(x) = 3x^4 - 4x^3$
$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$
$f'(x) = 0 \Rightarrow x = 0, x = 1.$

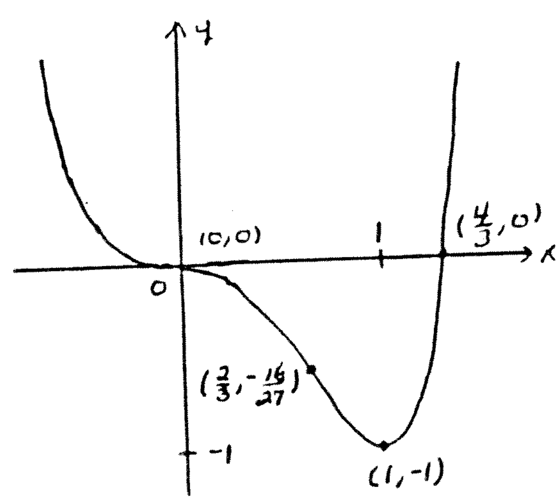


So $(1, -1)$ is a local minimum.

$$\begin{aligned} f''(x) &= 36x^2 - 24x \\ &= 12x(3x-2) \\ f''(x) &= 0 \Rightarrow x = 0, x = \frac{2}{3}. \end{aligned}$$

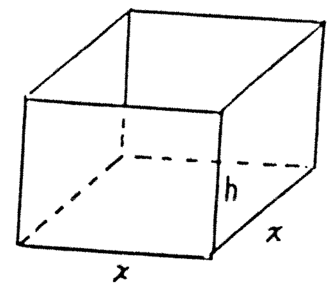
So, $(0,0)$ and $\left(\frac{2}{3}, \frac{-16}{27}\right)$ are inflection points.

Intercepts: $x = 0 \Rightarrow y = 0 \; (0,0)$
$y = 0 \Rightarrow x^3(3x-4) = 0$
$x = 0, x = \frac{4}{3}$
$(0,0), (\frac{4}{3}, 0)$



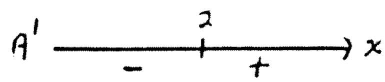
9. Let A represent the surface area to be minimized.

$$V = 4 \Rightarrow x^2h = 4 \Rightarrow h = \frac{4}{x^2}$$



$$\begin{aligned} A &= x^2 + 4xh \\ A(x) &= x^2 + 4x(\frac{4}{x^2}) = x^2 + \frac{16}{x} \text{ with } x > 0 \\ \text{Hence, } A'(x) &= 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2} = \frac{2(x-2)(x^2+2x+4)}{x^2} \end{aligned}$$

$A'(x) = 0$ when $x = 2$.



So, A has a minimum at $x = 2$.

Dimensions for least material are 2m by 2m by 1m.

10. (a) (i) $\int \left(\frac{1}{\sqrt[3]{x^2}} - 3x^{\frac{3}{2}} + \frac{1}{2x}\right) dx = \int \left(x^{-\frac{2}{3}} - 3x^{\frac{3}{2}} + \frac{1}{2}\cdot\frac{1}{x}\right) dx = 3x^{\frac{1}{3}} - \frac{6}{5}x^{\frac{5}{2}} + \frac{1}{2}\ln|x| + C$

(ii) $\int (3\cos 4x - 4e^{3x} - \sin 2x)dx = \frac{3}{4}\sin 4x - \frac{4}{3}e^{3x} + \frac{1}{2}\cos 2x + C$

(iii) $\int_1^5 \frac{1}{3x-2}dx = \frac{1}{3}\ln|3x-2| \Big|_1^5 = \frac{\ln 13}{3} - 0 = \frac{\ln 13}{3}$

(b) Points of intersection:

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = 2, x = -1$

$(2,-2), (-1,1)$

$$\begin{aligned} \text{Area} &= \int_{-1}^{2} [(2-x^2) - (-x)]dx \\ &= (2x - \frac{1}{3}x^3 + \frac{1}{2}x^2)\Big|_{-1}^{2} \\ &= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) \\ &= 6 - \frac{9}{3} + \frac{3}{2} = \frac{9}{2} \text{ sq. units} \end{aligned}$$