Ex 2) A rock is thrown upwards from a 200m cliff. If the height of the rock above the bottom of the cliff is given by $h(t) = -5t^2 + 20t + 200$, determine the maximum height of the rock.

$h' = -10t + 20$

$h' = 0 \Rightarrow -10t + 20 = 0$

$\underline{t = 2}$

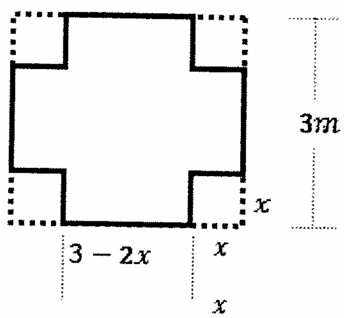
$h'' = -10 < 0$ $\forall$ values of t

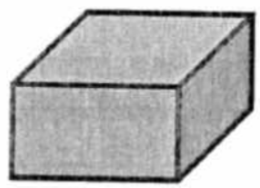
$\therefore$ max

$h(2) = -5(2)^2 + 20(2) + 200$

$= \underline{220 \text{ m.}}$

Ex 3) A box with an open top is to be constructed from a square piece of cardboard, 3ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest possible volume.





Done IN class

$3 - 2x$

Ex 4) Find two numbers whose difference is 100 and whose product is a minimum.

$x - y = 100$

$x = y + 100$

$P = xy$

$P = (y + 100)(y)$

$P = y^2 + 100y$

$P' = 2y + 100$

$P' = 0 \Rightarrow y = -50$

$P'' = 2 > 0$ $\forall$ values of x

$\therefore$ min

$y = -50$

$x = 50$

Mathematics 3208
More Optimization

Name: ____________________

1. A rectangular sheet of cardboard measures 24 cm by 18 cm. Congruent squares are cut from the corners of the sheet and the sides are folded to create a rectangular prism. Write a function to model the volume of the prism and use it to algebraically determine the maximum volume of the prism.

$x > 0$

$x < 9$

$V = x(24-2x)(18-2x)$

$V = x(432 - 84x + 4x^2)$

$V = 4x^3 - 84x^2 + 432x$

$V' = 12x^2 - 168x + 432$

$V' = 0 \Rightarrow 12(x^2 - 14x + 36) = 0$

$x = \dfrac{14 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)}$

$x \neq 10.6$ + 3.4

outside restriction

$A'' = 24x - 168$

$A''(3.4) = 24(3.4) - 168 = -8$

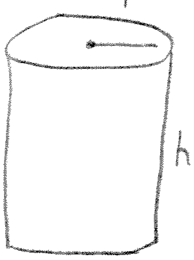
< 0 ∴ MAX

$V(3.4) = 4(3.4)^3 - 84(3.4)^2 + 432(3.4)$

$= 655 \text{ cm}^3$

2. An oil company wishes to reduce its costs by designing the most economical cylindrical barrels for shipping its oil. The industry standard uses 1.8 m^2 of material to manufacture each barrel. Using this amount of material, what dimensions would maximize the volume of the barrel?

r



h

$SA = 2\pi r^2 + 2\pi r h$

$1.8 = 2\pi r^2 + 2\pi r h$

$\dfrac{1.8 - 2\pi r^2}{2\pi r} = h$

$V = \pi r^2 h$

$V = \pi r^2 \left(\dfrac{1.8 - 2\pi r^2}{2\pi r}\right)$

$V = \dfrac{r}{2}(1.8 - 2\pi r^2)$

$V = .9r - \pi r^3$

$V' = .9 - 3\pi r^2$

$V' = 0 \Rightarrow 3\pi r^2 = .9$

$r = \sqrt{\dfrac{.9}{3\pi}}$

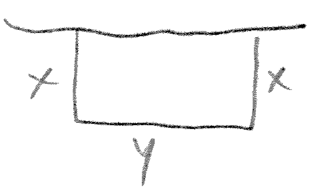
$r \approx .31$

$V'' = -6\pi r$

$V''(.31) = -5.8 < 0$ ∴ MAX

* $h = \dfrac{1.8 - 2\pi(.31)^2}{2\pi(.31)} = .61$

3. A farmer has 1000 metres of wooden fencing, and she wishes to fence off a rectangular plot of land bordered by a river, so that she does not require fencing on the side of the rectangle adjacent to the water. Find the dimensions of the rectangular plot with the greatest possible enclosed area.



$y = 1000 - 2x$

$A = x(1000 - 2x)$

$A = -2x^2 + 1000x$

$A' = -4x + 1000$

$A' = 0 \Rightarrow -4x + 1000 = 0$

$x = 250$

$A'' = -4$

$A''(250) = -4 < 0$

∴ MAX

$A(250) = -2(250)^2 + 1000(250)$

$= 375000 \text{ m}^2$

4. The sum of two non-negative numbers is 6. What is the ~~largest~~ SMALLEST possible value of the sum of their squares?

$x+y=6$

$y=6-x$

$x^2+(6-x)^2=P$

$x^2+36-12x+x^2=P$

$P=2x^2-12x+36$

$P'=4x-12$

$P'=0 \Rightarrow 4x-12=0$

$x=3$

$P(3)=2(3)^2-12(3)+36$

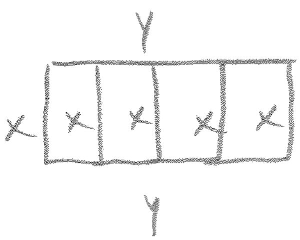
$P(3)=\underline{18}$

$P''=4$

$P''(3)=4>0$

$\therefore$ MIN

5. A farmer with 100 feet of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?



$5x+2y=100$

$y=50-\frac{5}{2}x$

$A=x\left(50-\frac{5}{2}x\right)$

$A=-\frac{5}{2}x^2+50x$

$A'=-5x+50$

$A'=0 \Rightarrow -5x+50=0$

$\boxed{x=10}$

$A''=-5$

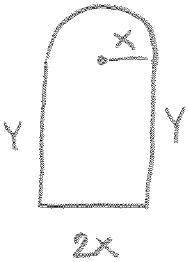
$A''(10)=-5<0$

$\therefore$ MAX

$A(10)=-\frac{5}{2}(10)^2+50(10)$

$=\underline{250\ ft^2}$

6. A Norman window consists of a rectangle with a semicircle mounted on top. Find the dimensions of the window with the largest area if its perimeter is 10 m.



Perimeter of $\frac{1}{2}$ circle

$10=2x+2y+\frac{2\pi x}{2}$

$10=2x+2y+\pi x$

$\therefore y=5-x-\frac{\pi}{2}x=\boxed{5-2.57x}$

$A=(2x)(y)+\frac{1}{2}\pi(x)^2$ → AREA of $\frac{1}{2}$ circle

$A=2x(5-2.57x)+1.57x^2$

$A=10x-5.14x^2+1.57x^2$

$A=-3.57x^2+10x$

$A'=-7.14x+10$

$A'=0 \Rightarrow \boxed{x=1.4}$

$A''=-7.14$

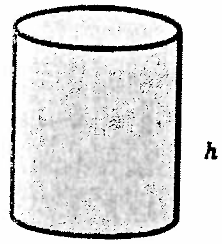
$A''(1.4)=-7.14<0$

$\therefore$ MAX

$\therefore A(1.4)=-3.57(1.4)^2+10(1.4)$

$=\underline{7\ m^2}$

Ex 5) A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.



$SA = 2\pi r^2 + 2\pi r h$

$SA = 2\pi r^2 + 2\pi r\left(\frac{1}{\pi r^2}\right)$

$SA = 2\pi r^2 + \frac{2}{r}$

$A' = 4\pi r - \frac{2}{r^2}$

$A' = 0 \Rightarrow 4\pi r = \frac{2}{r^2}$

$4\pi r^3 = 2$

$r^3 = \frac{1}{2\pi}$

$r = \sqrt[3]{\frac{1}{2\pi}} \approx 1.17$

$h = \frac{1}{\pi(1.17)^2} = .23$

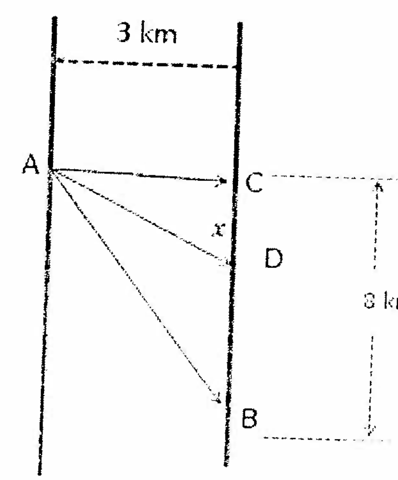
$V = \pi r^2 h$

$1 = \pi r^2 h$

$\frac{1}{\pi r^2} = h$

Ex 6) A man launches a boat from point A on a bank of a straight river, 3 km wide and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to point B, or he could row directly to point B. He could also row to any point D between C and B and then run to B from there. If he can row 6 km/hr and run 8 km/hr, where should he land his boat to reach point B as soon as possible? (assume current is neglible)

DONE IN Class



Ex 7) A school auditorium can seat 600 people at \$10 per ticket. The principal estimates that 50 less people wil attend if prices are increased by \$1. What ticket price would result in the greatest revenue? What is the maximum revenue possible?

$R = (600 - 50x)(10 + 1x)$

$R = 6000 + 100x - 50x^2$

$R' = -50x + 100$

$R' = 0 \Rightarrow x = 10$

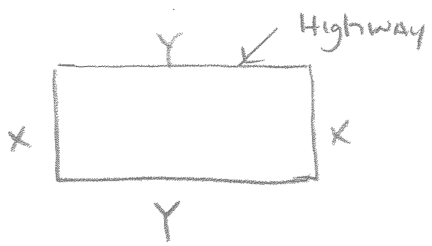
$R'' = -50 < 0 \quad \forall$ values of x

$\therefore$ MAX

$R(10) = 6000 + 100(10) - 50(10)^2$

$R = \$2000$

Ex 8) A rectangular lot adjacent to a highway is to be enclosed by a fence. I fencing cost \$6 per metre along the highway and \$4 per metre for the other three sides, find the largest area that can be enclosed for \$800.



$A = x\,y$

$A = x\left(200 - \frac{7}{2}x\right)$

$A = 200x - \frac{7}{2}x^2$

$A'' = -7 < 0 \quad \forall$ values of x

$\therefore$ MAX

$6x + 4x + 4y + 4x = 800$

$14x + 4y = 800$

$4y = 800 - 14x$

$y = 200 - \frac{7}{2}x$

$A' = 0 \Rightarrow -7x + 200 = 0$

$x = \frac{200}{7} \simeq 28.57$

$A(28.57) = 11428\ m^2$