MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION (Solutions) **Mathematics 1000** **WINTER 2011**

Marks

[12] 1. Using methods learned in this course, evaluate the following limits, showing your work.

a) $\lim\limits_{x \to -2} \dfrac{x^2 - 2x - 8}{x^2 + 5x + 6} = \lim\limits_{x \to -2} \dfrac{(x-4)(x+2)}{(x+3)(x+2)} = \lim\limits_{x \to -2} \dfrac{x-4}{x+3} = \dfrac{-2-4}{-2+3} = -6.$

b) $\lim\limits_{x \to 3} \dfrac{4 - \sqrt{x^2+7}}{x^2 - 9} = \lim\limits_{x \to 3} \left[\dfrac{4 - \sqrt{x^2+7}}{x^2-9} \cdot \dfrac{4 + \sqrt{x^2+7}}{4 + \sqrt{x^2+7}} \right] = \lim\limits_{x \to 3} \dfrac{16 - (x^2+7)}{(x^2-9)(4 + \sqrt{x^2+7})}$

$= \lim\limits_{x \to 3} \dfrac{16 - (x^2+7)}{(x^2-9)(4 + \sqrt{x^2+7})} = \lim\limits_{x \to 3} \dfrac{-(x^2-9)}{(x^2-9)(4 + \sqrt{x^2+7})} = \lim\limits_{x \to 3} \dfrac{-1}{4 + \sqrt{x^2+7}} = \dfrac{-1}{4 + \sqrt{16}} = -\dfrac{1}{8}.$

c) $\lim\limits_{x \to 7^+} \dfrac{|7 - x|}{x^2 - 49}$, $x \to 7^+ \Rightarrow x > 7 \Rightarrow x - 7 > 0 \Rightarrow 7 - x < 0$, $\therefore |7 - x| = -(7 - x)$.

$\lim\limits_{x \to 7^+} \dfrac{|7-x|}{x^2-49} = \lim\limits_{x \to 7^+} \dfrac{-(7-x)}{(x-7)(x+7)} = \lim\limits_{x \to 7^+} \dfrac{1}{x+7} = \dfrac{1}{14}.$

d) $\lim\limits_{x \to 0} \dfrac{\sin^2(5x)}{x^2 \cos^2(2x)} = \left[\lim\limits_{x \to 0} \dfrac{\sin(5x)}{x} \cdot \dfrac{5}{5} \right]^2 \cdot \lim\limits_{x \to 0} \dfrac{1}{\cos^2(2x)} = \left[5 \cdot \lim\limits_{x \to 0} \dfrac{\sin(5x)}{5x} \right]^2 \cdot \dfrac{1}{(\cos 0)^2} = [5 \cdot 1]^2 \cdot \dfrac{1}{1} = 25.$

[6] 2. Let $f(x) = \begin{cases} x+1, & \text{for } x < 1 \\ 2, & \text{for } 1 \le x \le 2 \\ x-1, & \text{for } x > 2 \end{cases}.$

Using the definition of continuity, determine all values at which $f(x)$ is discontinuous. Classify any discontinuities as removable or non-removable.

$f(x)$ is defined to be continuous at a point $x = a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

The function f has the possibility of discontinuities at $x = 1$ and 2.

At $x = 1$, $f(1) = 2$ by definition of f and $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x+1) = 1 + 1 = 2$. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 2 = 2$. Therefore, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 2 = f(1)$. Hence f is continuous at $x = 1$.

At $x = 2$, $f(2) = 2$ by definition of f and $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 2 = 2$. $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x-1) = 2 - 1 = 1$. Therefore, $\lim\limits_{x \to 2} f(x)$ does not exist and so f is not continuous at $x = 2$.

Since $\lim\limits_{x \to 2} f(x)$ does not exist, the discontinuity at $x = 2$ is nonremovable.

[7] 3. Use the DEFINITION OF DERIVATIVE to find $f'(x)$ for $f(x) = \sqrt{3x-4}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{3(x+h)-4}-\sqrt{3x-4}}{h}\cdot\frac{\sqrt{3(x+h)-4}+\sqrt{3x-4}}{\sqrt{3(x+h)-4}+\sqrt{3x-4}}$$

$$= \lim_{h\to 0}\frac{3(x+h)-4-(3x-4)}{h(\sqrt{3(x+h)-4}+\sqrt{3x-4})} = \lim_{h\to 0}\frac{3x+3h-4-3x+4}{h(\sqrt{3(x+h)-4}+\sqrt{3x-4})}$$

$$= \lim_{h\to 0}\frac{3h}{h(\sqrt{3(x+h)-4}+\sqrt{3x-4})} = \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)-4}+\sqrt{3x-4}}$$

$$= \frac{3}{\sqrt{3(x+0)-4}+\sqrt{3x-4}} = \frac{3}{\sqrt{3x-4}+\sqrt{3x-4}} = \frac{3}{2\sqrt{3x-4}}.$$

4. Differentiate each function and make any appropriate simplifications.

[4] a) $y = \tan^3\left(\sqrt{3x^2-2x}\right)$ DO NOT USE LOGARITHMIC DIFFERENTIATION

$$y' = 3\tan^2\left(\sqrt{3x^2-2x}\right)\cdot\sec^2\left(\sqrt{3x^2-2x}\right)\cdot\frac{6x-2}{2\sqrt{3x^2-2x}} = \frac{3(3x-1)\tan^2\left(\sqrt{3x^2-2x}\right)\cdot\sec^2\left(\sqrt{3x^2-2x}\right)}{\sqrt{3x^2-2x}}.$$

[3] b) $y = \dfrac{x^2-6}{(4x-3)^3}$ DO NOT USE LOGARITHMIC DIFFERENTIATION

$$y' = \frac{2x(4x-3)^3-3(4x-3)^2\cdot 4\cdot(x^2-6)}{(4x-3)^6} = \frac{2(4x-3)^2[x(4x-3)-6(x^2-6)]}{(4x-3)^6}$$

$$= \frac{2(4x^2-3x-6x^2+36)}{(4x-3)^4} = \frac{2(-2x^2-3x+36)}{(4x-3)^4} = \frac{-2(2x^2+3x-36)}{(4x-3)^4}$$

[4] c) $y = (e^{x^2})(\cot 6x^3)$ DO NOT USE LOGARITHMIC DIFFERENTIATION

$$y' = e^{x^2}\cdot 2x\cdot\cot 6x^3 + -\csc^2 6x^3\cdot 18x^2\cdot e^{x^2} = 2xe^{x^2}(\cot 6x^3 - 9x\csc^2 6x^3).$$

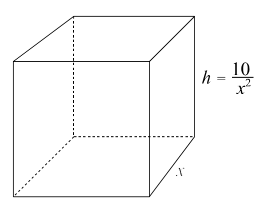
[6] d) $y = (\sin x)^{x^2} \Rightarrow \ln y = \ln(\sin x)^{x^2} = x^2\ln(\sin x).$

$$\frac{1}{y}\cdot y' = 2x\cdot\ln(\sin x) + \frac{1}{\sin x}\cdot\cos x\cdot x^2 = 2x\cdot\ln(\sin x) + x^2\cot x$$

$$y' = xy[2\ln(\sin x) + x\cot x] = x(\sin x)^{x^2}[2\ln(\sin x) + x\cot x].$$

[6] 5. A closed box with a square base is to have a volume of 10 m^3. The base costs $4/m^2$, the sides cost $2/m^2$ and the top $1/m^2$. What dimensions will give the minimum cost to build the box?

To find the cost of the box, we need to find the surface area. Given that the volume of the box is 10 m^3 and $V = x^2\cdot h$ we have $h = 10/x^2$.
Surface area is given by $S = x^2 + x^2 + 4xh = x^2 + x^2 + 4x\left(10/x^2\right)$. The terms of this expression are top, bottom and sides respectively of the box.
Therefore, the cost of the box is given by $C(x) = x^2 + 4x^2 + 4\cdot 2x\cdot\dfrac{10}{x^2}$.



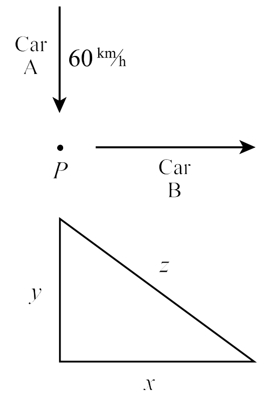
i.e. $C(x) = 5x^2 + \dfrac{80}{x} \Rightarrow C'(x) = 10x - \dfrac{80}{x^2} \Rightarrow C''(x) = 10 + \dfrac{160}{x^3}$.

$$C'(x) = 0 \Rightarrow 10x - \frac{80}{x^2} = 0 \Rightarrow 10x = \frac{80}{x^2} \Rightarrow 10x^3 = 80 \Rightarrow x^3 = 8 \Rightarrow x = 2.$$

$C''(2) = 10 + \dfrac{160}{8} = 10 + 20 > 0$ and so C is a minimum at $x = 2$.

The dimensions that allow for minimum cost are $(2 \times 2 \times 5/2)$ m.

[7] 6. Car A is being driven south toward point P at a speed of 60 km/h. Car B is being driven to the east away from point P. When car A is 0.6 km from point P, car B is 0.8 km from point P and the stright line distance between them is increasing at 20 km/h. What is the speed of car B?



Referring to the diagram at the right we are given $\frac{dy}{dt} = -60\ \frac{\text{km}}{\text{h}}$ and $\frac{dz}{dt} = 20\ \frac{\text{km}}{\text{h}}$, and asked to find $\frac{dx}{dt}$ when $x = 0.8$ km and $y = 0.6$ km.

The cardinal directions are perpendicular so we can use the Pythagorean Theorem.

$$x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \Rightarrow \frac{dx}{dt} = \frac{z\frac{dz}{dt} - y\frac{dy}{dt}}{x}.$$

$x = 0.8$ km and $y = 0.6$ km will mean that $z = 1$ km.

Hence, $\frac{dx}{dt} = \frac{z\frac{dz}{dt} - y\frac{dy}{dt}}{x} = \frac{1(20) - 0.6(-60)}{0.8} = \frac{20 + 36}{0.8} = \frac{56}{0.8} = 70\ \frac{\text{km}}{\text{h}}.$

Therefore, car B is travelling east at a speed of 70 km/h.

7. Find each of the following integrals.

[3] a) $\int [\sin(3x-2) + e^{2x}]\,dx = -\frac{1}{3}\cos(3x-2) + \frac{1}{2}e^{2x} + C.$

[3] b) $\int \left[\frac{(x+2)^2}{x}\right] dx = \int \left[\frac{x^2 + 4x + 4}{x}\right] dx = \int \left[x + 4 + \frac{4}{x}\right] dx = \frac{x^2}{2} + 4x + 4\ln|x| + C.$

[4] 8. a) Use implicit differentiation to find y' for $3x^2 + 2xy^2 + y^3 - 19 = 0$.

$$3x^2 + 2xy^2 + y^3 - 19 = 0 \Rightarrow 6x + 2y^2 + 4xyy' + 3y^2y' = 0 \Rightarrow 4xyy' + 3y^2y' = -6x - 2y^2$$

$$\Rightarrow y'[y(4x+3y)] = -2(3x+y^2) \Rightarrow y' = \frac{-2(3x+y^2)}{y(4x+3y)}.$$

[3] b) Use y' from part a) to find the equation of the tangent line at $(1, 2)$.

$$m_t \text{ at } (1,2) = \frac{-2(3(1)+2^2)}{2(4(1)+3(2))} = \frac{-(3+4)}{4+6} = -\frac{7}{10}.$$

The equation of the tangent to the curve at $(1, 2)$ will be given by

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = -\frac{7}{10}(x-1) \Rightarrow y - 2 = -\frac{7}{10}x + \frac{7}{10}$$

$$\Rightarrow y = -\frac{7}{10}x + \frac{7}{10} + \frac{20}{10} \Rightarrow y = -\frac{7}{10}x + \frac{27}{10}.$$

9. Given the following: $f(x) = \frac{x+1}{(x-1)^2}$, $f'(x) = \frac{-(x+3)}{(x-1)^3}$ and $f''(x) = \frac{2x+10}{(x-1)^4}$.

[3] a) Find the vertical asymptotes of $f(x)$, if any.

$$\lim_{x \to 1^-} \frac{x+1}{(x-1)^2} = \frac{1+1}{(1^- - 1)^2} = \frac{2}{(0^-)^2} = \frac{2}{0^+} \to +\infty$$

$$\lim_{x \to 1^+} \frac{x+1}{(x-1)^2} = \frac{1+1}{(1^+ - 1)^2} = \frac{2}{(0^+)^2} = \frac{2}{0^+} \to +\infty$$

Therefore, we have vertical asymptote $x = 1$.

[3] 9. b) Find the horizontal asymptotes of $f(x)$, if any.

$$\lim_{x \to \pm\infty} \frac{x+1}{(x-1)^2} = \lim_{x \to \pm\infty} \frac{x+1}{x^2 - 2x + 1} = \lim_{x \to \pm\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} = \frac{0}{1} = 0.$$

Hence the horizontal asymptote is $y = 0$ (the x-axis).

[2] c) Find the x- and y-intercepts of the graph of $f(x)$, if any.

To find the x-intercept we let $y = 0$. $\frac{x+1}{(x-1)^2} = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$. i.e. the x-intercept is -1.

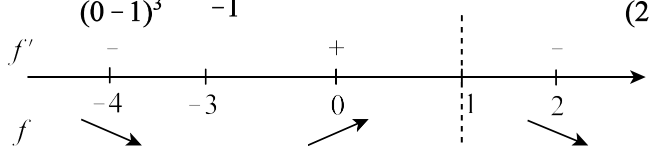
To find the y-intercept we let $x = 0$. $y = \frac{0+1}{(0-1)^2} \Rightarrow y = \frac{1}{1} = 1$. i.e. the y-intercept is 1.

[3] d) Determine the intervals on which $f(x)$ is increasing or decreasing and classify any relative (local) extrema.

$f'(x) = \frac{-(x+3)}{(x-1)^3} \Rightarrow f'(x) = 0$ when $x = -3$ and $f'(x)$ is undefined when $x = 1$.

On the interval $(-\infty, -3)$ (choose $x = -4$), $f'(-4) = \frac{-(-4+3)}{(-4-1)^3} = \frac{1}{-125} < 0$.

On $(-3, 1)$ ($x = 0$), $f'(0) = \frac{-(0+3)}{(0-1)^3} = \frac{-3}{-1} > 0$. On $(1, \infty)$ ($x = 2$), $f'(2) = \frac{-(2+3)}{(2-1)^3} = \frac{-5}{1} < 0$.



Therefore, f is decreasing on $(-\infty, -3)$ and $(1, \infty)$ and increasing on $(-3, 1)$.
We have already determined that there is a vertical asymptote at $x = 1$. Now we note that

$$f(-3) = \frac{-3+1}{(-3-1)^2} = \frac{-2}{16} = -\frac{1}{8}.$$

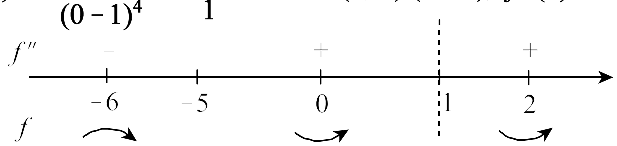
Hence, $(-3, -\frac{1}{8})$ is a relative minimum.

[3] e) Determine the intervals on which $f(x)$ is concave up or concave down and identify any inflection points.

$f''(x) = \frac{2x+10}{(x-1)^4}$. Clearly, $f''(x) = 0$ when $x = -5$ and $f''(x)$ is undefined when $x = 1$.

On the interval $(-\infty, -5)$ (choose $x = -6$), $f''(-6) = \frac{2(-6)+10}{(-6-1)^4} = \frac{-2}{2401} < 0$.

On $(-5, 1)$ ($x = 0$), $f''(0) = \frac{2(0)+10}{(0-1)^4} = \frac{10}{1} > 0$. On $(1, \infty)$ ($x = 2$), $f''(2) = \frac{2(2)+10}{(2-1)^4} = \frac{14}{1} > 0$.

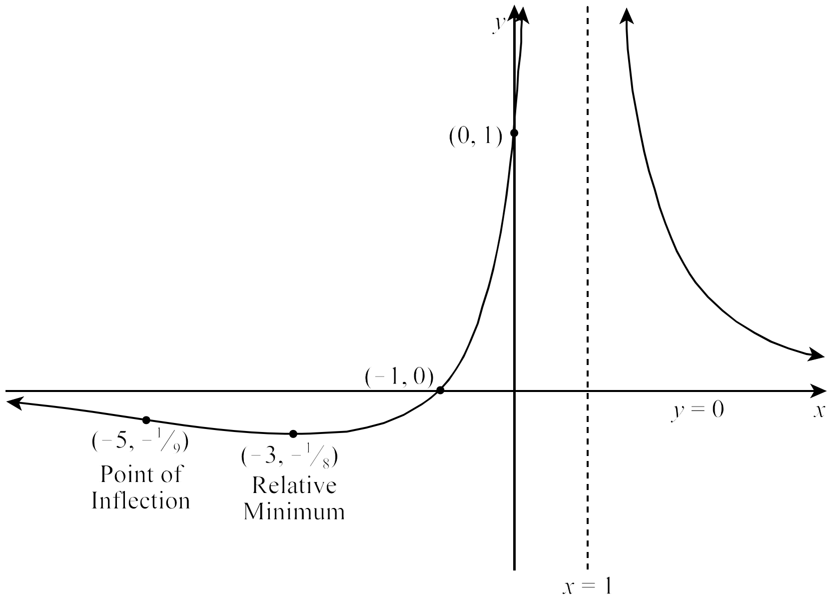


Therefore, f is concave down on $(-\infty, -5)$ and concave up on $(-5, 1)$ and $(1, \infty)$ and so there is a point of inflection at $x = -5$.

$$f(-5) = \frac{-5+1}{(-5-1)^2} = \frac{-4}{36} = -\frac{1}{9}.$$

Hence, $(-5, -\frac{1}{9})$ is the point of inflection.

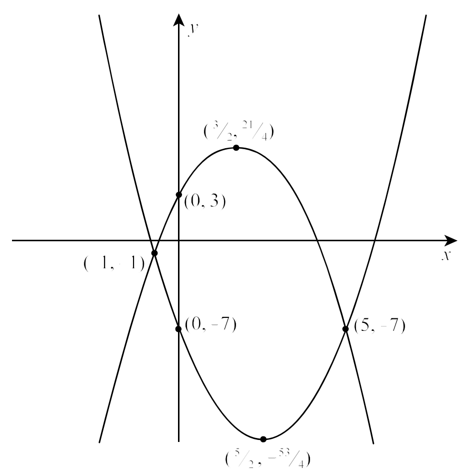
[3] 9. f) Sketch the graph of $f(x)$. Label any inflection points and extrema.



[7] 10. Find the area bounded by the graphs of $y = x^2 - 5x - 7$ and $y = 3x + 3 - x^2$.

$y = x^2 - 5x - 7 = x^2 - 5x + \frac{25}{4} - 7 - \frac{25}{4} = \left(x - \frac{5}{2}\right)^2 - \frac{53}{4}$ is a parabola opening upward from $(5/2, -53/4)$ and having y-intercept $(0, -7)$.

$y = 3x + 3 - x^2 = -(x^2 - 3x) + 3 = -\left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4} + 3 = -\left(x - \frac{3}{2}\right)^2 + \frac{21}{4}$ is a parabola opening downward from $(3/2, 21/4)$ with y-intercept $(0, 3)$.



To find points of intersection of the parabolae and hence the limits of integration we equate the y's and solve.

$$x^2 - 5x - 7 = 3x + 3 - x^2 \Rightarrow 2x^2 - 8x - 10 = 0$$
$$\Rightarrow 2(x^2 - 4x - 5) = 0 \Rightarrow 2(x - 5)(x + 1) = 0$$
$$\Rightarrow x = 5 \text{ or } x = -1$$

Taking $x = 0$ from the interval $(-1, 5)$ we see that $3x + 3 - x^2$ has value 3 and $x^2 - 5x - 7$ has value -7. Thus the area of the region will be given by:

$$A = \int_{-1}^{5} [(3x + 3 - x^2) - (x^2 - 5x - 7)]\, dx$$
$$= \int_{-1}^{5} (3x + 3 - x^2 - x^2 + 5x + 7)\, dx$$
$$= \int_{-1}^{5} (10 + 8x - 2x^2)\, dx$$
$$= \left(10x + 4x^2 - \frac{2x^3}{3}\right)\Big|_{-1}^{5}$$
$$= 10 \cdot 5 + 4 \cdot 5^2 - \frac{2 \cdot 5^3}{3} - \left(10(-1) + 4(-1)^2 - \frac{2(-1)^3}{3}\right)$$
$$= 50 + 100 - \frac{250}{3} + 10 - 4 - \frac{2}{3}$$
$$= 156 - \frac{252}{3} = 156 - 84 = 72.$$

Thus, the area bounded by the curves is 72 units^2.

[8] 11. Answer ONE of the following:

a) Prove that if $H(x) = f(x) - g(x)$ then $H'(x) = f'(x) - g'(x)$.

Proof:

$$H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - g(x+h) - [f(x) - g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) - [g(x+h) - g(x)]}{h} = \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) - g'(x) \square$$

OR

b) Given that $xy + y^2 = 1$, show that $y'' = \dfrac{2}{(x+2y)^3}$.

$$xy + y^2 = 1 \Rightarrow 1 \cdot y + xy' + 2yy' = 0 \Rightarrow xy' + 2yy' = -y \Rightarrow y'(x+2y) = -y \Rightarrow y' = \frac{-y}{x+2y}.$$

$$y'' = \frac{-y'(x+2y) - (1+2y')(-y)}{(x+2y)^2} = \frac{-xy' - 2yy' + y + 2yy'}{(x+2y)^2} = \frac{-xy' + y}{(x+2y)^2} = \frac{-x\left(\frac{-y}{x+2y}\right) + y}{(x+2y)^2}$$

$$= \frac{xy + xy + 2y^2}{x+2y} \cdot \frac{1}{(x+2y)^2} = \frac{2xy + 2y^2}{(x+2y)^3} = \frac{2(xy + y^2)}{(x+2y)^3} = \frac{2(1)}{(x+2y)^3} = \frac{2}{(x+2y)^3}.$$