MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION Solutions **Mathematics 1000** **FALL 2010**

Marks

[12] 1. Evaluate the following limits, showing your work. Assign ∞ or $-\infty$ as appropriate.

a) $$\lim_{x\to 3}\frac{x^2+2x-15}{2x^2-5x-3} = \lim_{x\to 3}\frac{(x+5)(x-3)}{(2x+1)(x-3)} = \lim_{x\to 3}\frac{x+5}{2x+1} = \frac{3+5}{6+1} = \frac{8}{7}.$$

b) $$\lim_{x\to -1}\frac{4x+4}{2x+\sqrt{x^2+3}} = \lim_{x\to -1}\frac{4x+4}{2x+\sqrt{x^2+3}}\cdot\frac{2x-\sqrt{x^2+3}}{2x-\sqrt{x^2+3}} = \lim_{x\to -1}\frac{4(x+1)(2x-\sqrt{x^2+3})}{4x^2-(x^2+3)}$$
$$= \lim_{x\to -1}\frac{4(x+1)(2x-\sqrt{x^2+3})}{4x^2-x^2-3} = \lim_{x\to -1}\frac{4(x+1)(2x-\sqrt{x^2+3})}{3x^2-3}$$
$$= \lim_{x\to -1}\frac{4(x+1)(2x-\sqrt{x^2+3})}{3(x^2-1)} = \lim_{x\to -1}\frac{4(x+1)(2x-\sqrt{x^2+3})}{3(x+1)(x-1)}$$
$$= \lim_{x\to -1}\frac{4(2x-\sqrt{x^2+3})}{3(x-1)} = \frac{4(-2-\sqrt{4})}{3(-2)} = \frac{4(-4)}{-6} = \frac{-16}{-6} = \frac{8}{3}.$$

c) $$\lim_{x\to 0}\frac{\sin^2(4x)}{x^2} = \left[\lim_{x\to 0}\frac{\sin(4x)}{x}\right]^2 = \left[\lim_{x\to 0}\frac{\sin(4x)}{x}\cdot\frac{4}{4}\right]^2 = \left[4\lim_{x\to 0}\frac{\sin(4x)}{4x}\right]^2 = [4\cdot 1]^2 = 16$$

[8] 2. Given the function $f(x) = \begin{cases}\dfrac{6}{1-x}, & \text{for } x \le -1\\ \dfrac{x-2}{x^2-4}, & \text{for } x > -1\end{cases}$, using the definition of continuity, determine all points at which $f(x)$ is not continuous. Classify any discontinuities as removable or non-removable.

We define function f to be continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$.

The function f could have discontinuities at $x = -1$ and $x = 2$.

Now, $f(-1) = \dfrac{6}{1+1} = \dfrac{6}{2} = 3$ and so f is defined at $x = -1$. However, $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-}\dfrac{6}{1-x} = \dfrac{6}{2} = 3$, while $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+}\dfrac{x-2}{x^2-4} = \dfrac{-1-2}{1-4} = \dfrac{-3}{-3} = 1$. Therefore, $\lim_{x\to -1} f(x)$ does not exist. Hence, f is discontinuous at $x = -1$ and the discontinuity is non-removable.

$f(2)$ is undefined because the denominator is 0 and so f is also discontinuous at $x = 2$. However, $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\dfrac{x-2}{(x-2)(x+2)} = \lim\limits_{x\to 2}\dfrac{1}{x+2} = \dfrac{1}{4}$. Since $\lim_{x\to 2} f(x)$ exists, the discontinuity at $x = 2$ is removable.

[10] 3. a) State the limit definition of the derivative of a function $f(x)$.

The derivative of $f(x)$ is defined by $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

3. b) Use the limit definition to differentiate $f(x) = \dfrac{3x+1}{2-x}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left[\frac{3(x+h)+1}{2-(x+h)} - \frac{3x+1}{2-x}\right]\cdot\frac{1}{h}$$
$$= \lim_{h\to 0}\left[\frac{(3x+3h+1)(2-x)-(3x+1)(2-x-h)}{(2-x-h)(2-x)}\right]\cdot\frac{1}{h}$$
$$= \lim_{h\to 0}\left[\frac{6x+6h+2-3x^2-3xh-x-(6x-3x^2-3xh+2-x-h)}{(2-x-h)(2-x)}\right]\cdot\frac{1}{h}$$
$$= \lim_{h\to 0}\frac{6x+6h+2-3x^2-3xh-x-6x+3x^2+3xh-2+x+h}{h(2-x-h)(2-x)}$$
$$= \lim_{h\to 0}\frac{7h}{h(2-x-h)(2-x)} = \lim_{h\to 0}\frac{7}{(2-x-h)(2-x)} = \frac{7}{(2-x)^2}.$$

c) Verify your answer to part b) using the quotient rule.

$$f(x) = \frac{3x+1}{2-x} \Rightarrow f'(x) = \frac{3(2-x)+(3x+1)}{(2-x)^2} = \frac{6-3x+3x+1}{(2-x)^2} = \frac{7}{(2-x)^2}.$$

[16] 4. Compute $\frac{dy}{dx}$, making any obvious simplifications.

a) $y = x^5 e^{x^5} \Rightarrow y' = 5x^4\cdot e^{x^5} + e^{x^5}\cdot 5x^4\cdot x^5 = 5x^4 e^{x^5}[1+x^5]$

b) $y = \ln\left(\dfrac{\sqrt{x}}{e^x\tan^3 x}\right) = \ln\sqrt{x} - \ln e^x - \ln(\tan^3 x) = \dfrac{1}{2}\ln x - x - 3\ln(\tan x)$

$y' = \dfrac{1}{2}\cdot\dfrac{1}{x} - 1 - 3\cdot\dfrac{1}{\tan x}\cdot\sec^2 x = \dfrac{1}{2x} - 1 - \dfrac{3\sec^2 x}{\tan x}.$

c) $y = \sin^7\sqrt{x} \Rightarrow y' = 7\sin^6\sqrt{x}\cdot\cos\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{7\sin^6\sqrt{x}\,\cos\sqrt{x}}{2\sqrt{x}}$

d) $y = (\sec x)^{4x} \Rightarrow \ln y = \ln(\sec x)^{4x} = 4x\cdot\ln(\sec x)$

$\dfrac{1}{y}\cdot y' = 4\cdot\ln\sec x + \dfrac{1}{\sec x}\cdot\sec x\tan x\cdot 4x = 4\ln\sec x + 4x\tan x$

$y' = (\sec x)^{4x}[4\ln\sec x + 4x\tan x] = 4(\sec x)^{4x}[\ln\sec x + x\tan x].$

[5] 5. Using Implicit differentiation, find the equation of the tangent line to the curve defined by the equation $\pi(x^2 - y^2) = \cos(\pi y)$ at the point $(3/2, 3/2)$.

$$\pi(x^2-y^2) = \cos(\pi y) \Rightarrow \pi(2x-2yy') = -\sin(\pi y)\cdot\pi y' \Rightarrow 2x - 2yy' = -y'\sin(\pi y)$$
$$\Rightarrow 2x = 2yy' - y'\sin(\pi y) \Rightarrow y'(2y - \sin(\pi y)) = 2x \Rightarrow y' = \frac{2x}{2y-\sin(\pi y)}$$

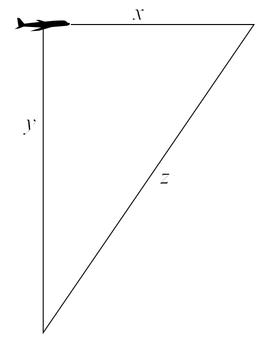
At $(3/2, 3/2)$, $y' = \dfrac{2(3/2)}{2(3/2) - \sin(\pi\cdot 3/2)} = \dfrac{3}{3-(-1)} = \dfrac{3}{4}.$

Using the point/slope form of the equation of a line we derive the equation of the required tangent

$$y - \frac{3}{2} = \frac{3}{4}\left(x - \frac{3}{2}\right) \Rightarrow y = \frac{3}{2} + \frac{3}{4}x - \frac{9}{8} \Rightarrow y = \frac{3}{4}x + \frac{3}{8}.$$

[8] 6. An airplane is flying due east at a constant altitude of 16 km, and passes directly over an air control tower. Later, the plane passes over a route marker which lies 12 km east of the tower. At this moment, the distance between the tower and the plane is measured to be changing at a rate of 400 km per hour. What is the speed of the plane at this time?

6. We are asked to determine the speed of the plane, $\frac{dx}{dt}$ when $y = 16$ and $x = 12$ km, given that $\frac{dz}{dt} = 400$ km per hour.



$$x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \Rightarrow \frac{dx}{dt} = \frac{z\frac{dz}{dt} - y\frac{dy}{dt}}{x}$$

Since the altitude is constant, $\frac{dy}{dt} = 0$ and we have $z^2 = 12^2 + 16^2 \Rightarrow z = 20$,

$$\frac{dx}{dt} = \frac{20(400) - 16(0)}{12} = \frac{8000}{12} = 666\frac{2}{3} \text{ kph.}$$

The speed of the plane at this time is $666\frac{2}{3}$ kph.

[15] 7. Consider the function $f(x) = \frac{4x}{(x-1)^2}$ with derivatives $f'(x) = \frac{-4(x+1)}{(x-1)^3}$ and $f''(x) = \frac{8(x+2)}{(x-1)^4}$.

a) Find any vertical asymptotes of the graph of f.

$\lim_{x \to 1^+} \frac{4x}{(x-1)^2} = \frac{4(1)}{0^+}$. We say that $f \to +\infty$ as $x \to 1$ from the right. Therefore, $x = 1$ is a vertical asymptote.

b) Find any horizontal asymptotes of the graph of f.

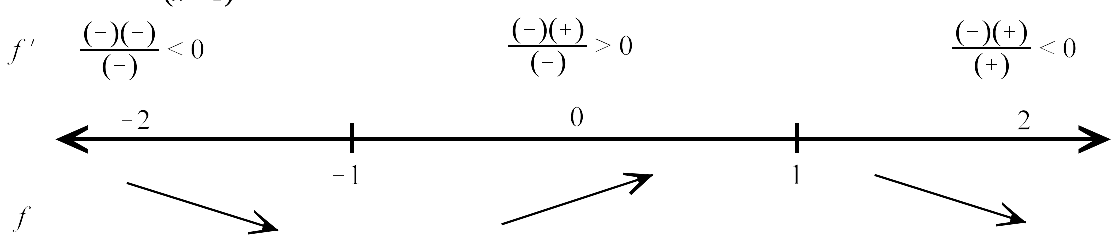
$\lim_{x \to \infty} \frac{4x}{(x-1)^2} = \lim_{x \to \infty} \frac{4x}{x^2 - 2x + 1} = \lim_{x \to \infty} \frac{\frac{4}{x}}{1 - \frac{2}{x} + \frac{1}{x^2}} = \frac{0}{1} = 0$. Since the calculation is the same for $x \to -\infty$, $y = 0$ is a vertical asymptote.

c) Find any x- and y-intercepts of the graph of f.

If $x = 0$ then $y = 0$, so 0 is the x- and y-intercept.

d) Find the intervals on which $f(x)$ is increasing or decreasing, and classify any relative (local) extrema.

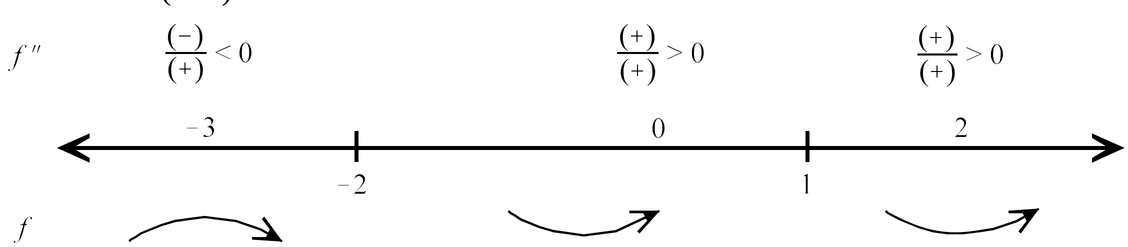
$f'(x) = 0 \Rightarrow \frac{-4(x+1)}{(x-1)^3} = 0 \Rightarrow -4(x+1) = 0 \Rightarrow x = -1$. Also, $f'(x)$ is undefined when $x = 1$.



We see that f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$. There is a relative minimum at $(-1\ -1)$.

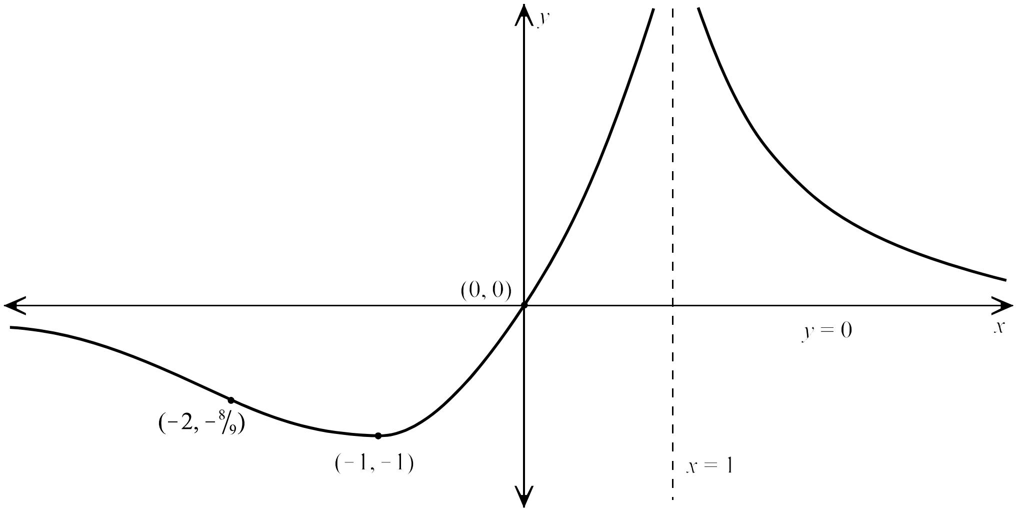
e) Determine the intervals on which the graph of f is concave upward or concave downward and identify any points of inflection.

$f''(x) = 0 \Rightarrow \frac{8(x+2)}{(x-1)^4} = 0 \Rightarrow 8(x+2) = 0 \Rightarrow x = -2$. Furthermore, f'' is undefined at $x = 1$.



Hence, f is concave down on $(-\infty, -2)$ and concave up on $(-2, 1)$ and $(1, \infty)$. There is a point of inflection at $(-2, -\frac{8}{9})$.

f) Sketch the graph of f. Label your graph carefully.



[12] 8. Find each of the following integrals.

a) $\displaystyle\int \frac{(x-2)^2}{x}\,dx = \int \frac{x^2-4x+4}{x}\,dx = \int\left[x-4+\frac{4}{x}\right]dx = \frac{x^2}{2} - 4x + 4\ln|x| + C.$

b) $\displaystyle\int \sqrt{5x+4}\,dx = \int (5x+4)^{1/2}\,dx = \frac{1}{5}\cdot\frac{2}{3}\cdot(5x+4)^{3/2} + C = \frac{2}{15}(5x+4)\sqrt{5x+4} + C.$

c) $\displaystyle\int_0^\pi \cos\frac{x}{6}\,dx = 6\sin\frac{x}{6}\Big]_0^\pi = 6\sin\frac{\pi}{6} - 6\sin 0 = 6\cdot\frac{1}{2} - 6\cdot 0 = 3.$

[6] 9. Indicate whether each of the following statements is true or false. If the statement is true, briefly explain why. If the statement is false, briefly explain why or give a counterexample that illustrates this.

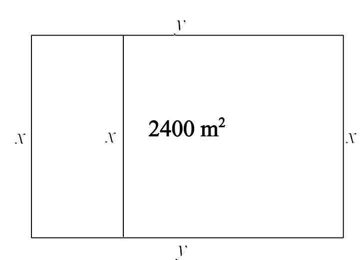
a) If a function f is continuous at a point, then it is also differentiable at that point.
False. $f(x) = |x|$ is continuous at $x = 0$, but $f'(0)$ does not exist. On $(-\infty, 0)$, $f'(x) = -1$, while on $(0, \infty)$, $f'(x) = 1$.

b) A function f defined on an open interval must achieve an absolute (global) maximum and an absolute (global) minimum value.
False. The function $f(x) = \tan x$ is defined on the open interval $(-\pi/2, \pi/2)$, but achieves neither an absolute minimum nor maximum on the interval.

[8] 10. Do **EITHER** part a) **OR** part b).

a) A farmer wants to fence a rectangular plot of land with area 2400 m^2. She wants to keep her horses on one side of the field, so she plans to use additional fencingto build an internal divider, parallel to two sides of the fence. Wood for the outer walls costs \$3 per metre, and wood for the internal divider costs \$2 per metre. What is the minimum cost of the project?

$xy = 2400 \Rightarrow y = \dfrac{2400}{x}$, The cost of the fence is given by $C(x)$ where



$$C(x) = 3(2x+2y) + 2x = 3\left(2x + 2\frac{2400}{x}\right) + 2x$$
$$= 6x + \frac{14400}{x} + 2x = 8x + \frac{14400}{x}$$

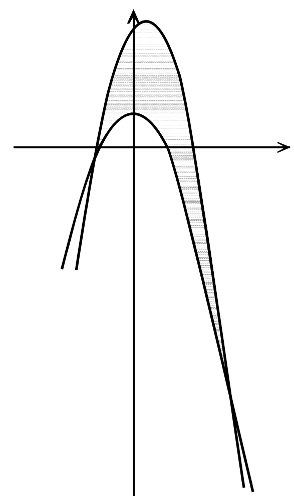
$C'(x) = 8 - \dfrac{14400}{x^2} = 8 - 14400x^{-2} \Rightarrow C''(x) = 28800x^{-3} = \dfrac{28800}{x^3}$.

$C'(x) = 0 \Rightarrow 8 - \dfrac{14400}{x^2} = 0 \Rightarrow 8x^2 = 14400 \Rightarrow x^2 = 1800 \Rightarrow x = 30\sqrt{2}$.

$C''(30\sqrt{2}) = \dfrac{28800}{(30\sqrt{2})^3} > 0$. And thus $x = 30\sqrt{2}$ gives a minimum for C. The actual minimum cost of the fencing is given by $C(30\sqrt{2}) = 8(30\sqrt{2}) + \dfrac{14400}{30\sqrt{2}} = 240\sqrt{2} + \dfrac{480}{\sqrt{2}} = 480\sqrt{2} \approx \$678.82$.

b) Find the area of the region enclosed by the curves $y = 1 - x^2$ and $y = (x + 1)(7 - 3x)$.

To determine the limits of integration we let $1 - x^2 = (x + 1)(7 - 3x)$.
$1 - x^2 = 7x - 3x^2 + 7 - 3x \Rightarrow 2x^2 - 4x - 6 = 0 \Rightarrow x^2 - 2x - 3 = 0$
$\Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = 3$ or $x = -1$.
On the intervall $[-1, 3]$, we consider $x = 0$ and find that $1 - x^2 = 1$ and $(x + 1)(7 - 3x) = 7$. Therefore, the area bounded by the curves is given by:



$$\begin{aligned}
A &= \int_{-1}^{3} [(x+1)(7-3x) - (1 - x^2)]\, dx \\
&= \int_{-1}^{3} [-3x^2 + 4x + 7 - 1 + x^2]\, dx \\
&= \int_{-1}^{3} [-2x^2 + 4x + 6]\, dx \\
&= -\frac{2}{3}x^3 + 2x^2 + 6x \Big]_{-1}^{3} \\
&= (-18 + 18 + 18) - \left(\frac{2}{3} + 2 - 6\right) \\
&= 18 - \frac{2}{3} + 4 = 22 - \frac{2}{3} = \frac{64}{3} \text{ square units.}
\end{aligned}$$