MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION | **Mathematics 1000** | WINTER 2013

COMPLETE THE FOLLOWING CAREFULLY AND CLEARLY:

(Please Print)

Surname: Solutions

Given Names: ______

MUN Number: ______

Instructor: Craighead Leonard Suvak Wang

Please note:

This exam has **EIGHT** pages of questions.

All calculators are strictly forbidden.

The questions are to be answered in the spaces provided.

Under no circumstances may the candidate take this book from the examination room.

On no account are pages to be torn or removed from this book, unless specifically directed.

Candidates must not have in their possession books, notes or papers of any kind, unless specifically directed.

No electronic devices of any kind, including cell phones and MP3 players, are permitted at your desk.

	MARKS
14	1.
3	2.
7	3.
5	4.
18	5.
9	6.
10	7.
6	8.
5	9.
8	10.
10	11.
5	12.
100	Total

FOR INSTRUCTOR'S USE ONLY

FINAL 55%	TERM 45%	TOTAL 100%	FINAL MARK	GRADE

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FINAL EXAMINATION | MATHEMATICS 1000 | Winter, 2013

Calculators are not permitted on this examination.

1. Evaluate each of the following limits, assigning ∞ or $-\infty$ where appropriate. You may not use L'Hospital's Rule.

[3] (a) $\lim\limits_{x\to 9}\dfrac{3-\sqrt{x}}{x^2-81}$

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}=\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\cdot\overbrace{\frac{3+\sqrt{x}}{3+\sqrt{x}}}^{1.5}=\lim_{x\to 9}\frac{9-x}{(x-9)(x+9)(3+\sqrt{x})}=\overbrace{\lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}}^{0.5}$$

$$=\underbrace{\frac{-1}{18\cdot 6}=-\frac{1}{108}}_{0.5}$$

[3] (b) $\lim\limits_{x\to-\infty}\dfrac{4x-3}{\sqrt{4x^2-9}}$

$$\lim_{x\to-\infty}\frac{4x-3}{\sqrt{4x^2-9}}=\overbrace{\lim_{x\to-\infty}\frac{\frac{1}{x}(4x-3)}{\frac{1}{x}\sqrt{4x^2-9}}=\lim_{x\to-\infty}\frac{4-\frac{3}{x}}{\cancel{\ominus}\sqrt{4-\frac{9}{x^2}}}}^{2}=\overbrace{\frac{4-0}{\cancel{\ominus}\sqrt{4-0}}=\frac{4}{\cancel{\ominus}2}}^{1}=\cancel{\ominus}2$$

OR $$\lim_{x\to-\infty}\frac{4x-3}{\sqrt{4x^2-9}}\approx\underbrace{\lim_{x\to-\infty}\frac{4x}{\sqrt{4x^2}}}_{2}=\underbrace{\lim_{x\to-\infty}\frac{4x}{2x}=4/2=2}_{1}$$

(c) Given $f(x)=\dfrac{x^3-125}{2x^3-50x}$, find

[3] (i) $\lim\limits_{x\to 5}f(x)$

$$\lim_{x\to 5}f(x)=\lim_{x\to 5}\frac{x^3-125}{2x^3-50x}=\overbrace{\lim_{x\to 5}\frac{(x-5)(x^2+5x+25)}{2x(x^2-5)}}^{1.5}=\overbrace{\lim_{x\to 5}\frac{(x-5)(x^2+5x+25)}{2x(x-5)(x+5)}}^{0.5}$$

$$=\underbrace{\lim_{x\to 5}\frac{x^2+5x+25}{2x(x+5)}=\frac{25+25+25}{10(10)}=\frac{75}{100}=\frac{3}{4}}_{1}$$

[3] (ii) $\lim\limits_{x\to-5^-}f(x)$ OR $\overbrace{\lim\limits_{x\to-5^-}\dfrac{x^3-125}{2x^3-50x}=\lim\limits_{x\to-5^-}\dfrac{\cancel{(x-5)}(x^2+5x+25)}{2x\cancel{(x-5)}(x+5)}=\%}^{1.5}$ ∴ Step #3

~~$\lim\limits_{x\to-5^-}f(x)=\lim\limits_{x\to-5^-}\dfrac{x^2+5x+25}{2x(x+5)}=\dfrac{25-25+25}{-10(0^-)}=\dfrac{25}{0+}=\infty$~~] OK but not taught.

Step #3: Know $+\infty$, $-\infty$ or DNE so test $x\to-5^-$, $x=-6$

$\dfrac{(-6)^3-125}{2(-6)^3-50(-6)}=\dfrac{-}{-}=\oplus$ ∴ $\boxed{+\infty}$ } 1.5

[2] (iii) $\lim\limits_{x\to\infty}f(x)$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{x^3-125}{2x^3-50x}=\underbrace{\frac{1}{2}}_{1.5}$$ $\underbrace{\text{As Power Top = Bottom}}_{0.5}$

[3] 2. Give the horizontal and vertical asymptotes, if any, for the graph of the function of #1(c).

Horizontal asymptote: $y=\frac{1}{2}$

Vertical asymptotes: $x=0$ and $x=-5$

HA

1 [$y=1/2$ As $\lim\limits_{x\to+\infty/-\infty}f(x)=1/2$ As in 1c(iii)

1/2 [Test (If crossed)

$\frac{1}{2}=\frac{x^3-125}{2x^3-50x}$

$\Rightarrow \cancel{2x^3}-250=\cancel{2x^3}-50x$

$\Rightarrow 50x=250$ ∴ $x=5$

→ Crossed at $x=5$

VA

1 [$2x^3-50x=0$

$2x(x^2-25)=0$

$2x(x+5)(x-5)=0$

$x=0,\ x=-5,\ x=5$

Test

$f(0)=\frac{-125}{0}$ ✓ VA

$f(-5)=\frac{-250}{0}$ ✓ VA

$f(5)=\frac{0}{0}$ ✗ Hole

∴ VA: $\underline{x=0,-5}$ 0.5

[7] 3. Let

$$f(x) = \begin{cases} 3a\cos x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ ae^x + 4 & \text{if} \quad x > 0 \end{cases}$$

(i) Find a so that $\lim\limits_{x\to 0} f(x)$ exists. (ii) Is f continuous at $x = 0$ for this value of a? (iii) If not, is the discontinuity at $x = 0$ removable. Verify your answers.

(i)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3a\cos x = 3a(1) = 3a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (ae^x + 4) = a(1) + 4 = a + 4$$

2.5

For $\lim\limits_{x\to 0} f(x)$ to exist we must have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$3a = a + 4$$

$$a = 2$$

~~Then $\lim\limits_{x\to 0^-} f(x) = 3(2) = 6$ and $\lim\limits_{x\to 0^+} f(x) = 2 + 4 = 6$. So $\lim\limits_{x\to 0} f(x) = 6$. Now $f(0) = 0$. So $\lim\limits_{x\to 0} f(x) \neq f(0)$. So f is not continuous at $x = 0$ for any a. The discontinuity at $x = 0$ is removable since f can be made continuous at $x = 0$ by redefining $f(0) = 6$.~~

(ii) CONTINUOUS AT $X=0$ WHEN $a=2$:

$\lim\limits_{x\to 0^-} 3(2)\cos x = \lim\limits_{x\to 0^-} 6\cos x = 6$

$\lim\limits_{x\to 0^+} (2e^x + 4) = 2e^0 + 4 = 6$

$f(0) = 0$

DON'T MATCH ∴ NOT CONTINUOUS (DISCONTINUOUS)

2.5

(iii) DISCONT. AT $X=0$ REMOVABLE AS $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = \underline{6}$

2

[5] 4. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{ax+b}$, where a and b are constants.

$$f'(x) = \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{1} = \underbrace{\lim_{h\to 0} \frac{\sqrt{a(x+h)+b} - \sqrt{ax+b}}{h}}_{0.5}$$

$$= \lim_{h\to 0} \frac{\sqrt{a(x+h)+b} - \sqrt{ax+b}}{h} \cdot \underbrace{\frac{\sqrt{a(x+h)+b} + \sqrt{ax+b}}{\sqrt{a(x+h)+b} + \sqrt{ax+b}}}_{1}$$

$$= \lim_{h\to 0} \frac{a(x+h) + b - (ax+b)}{h(\sqrt{a(x+h)+b} + \sqrt{ax+b})} = \lim_{h\to 0} \frac{ax + ah + b - ax - b}{h(\sqrt{a(x+h)+b} + \sqrt{ax+b})}$$

$$= \lim_{h\to 0} \frac{ah}{h(\sqrt{a(x+h)+b} + \sqrt{ax+b})} = \lim_{h\to 0} \frac{a}{\sqrt{a(x+h)+b} + \sqrt{ax+b}} \quad \Big\} \ 1.5$$

$$= \frac{a}{\sqrt{ax+b} + \sqrt{ax+b}} = \frac{a}{2\sqrt{ax+b}} \quad \Big\} \ 1$$

5. Find and simplify the derivative of each of the following:

[3] (a) $f(x) = \dfrac{3}{1+\sin^3 2x}$

$$f'(x) = \underbrace{\frac{-3}{(1+\sin^3 2x)^2}\cdot 3\sin^2 2x\cos 2x\cdot 2}_{2.5} = \underbrace{\frac{-18\sin^2 2x\cos 2x}{(1+\sin^3 2x)^2}}_{0.5}$$

[5] (b) $f(x) = \dfrac{1+2\ln(\ln x)}{\ln^2 x}$

$$f'(x) = \overbrace{\frac{\underbrace{\ln^2 x\cdot\frac{2}{\ln x}\cdot\frac{1}{x}}_{1.5*} - \underbrace{[1+2\ln(\ln x)]\cdot 2\ln x\cdot\frac{1}{x}}_{1.5*}}{(\ln^2 x)^2}}^{1\ \text{QUOT. RULE} + 3^{*}\ \text{DER.}} = \overbrace{\frac{\frac{2\ln x}{x} - \frac{2[1+2\ln(\ln x)]\ln x}{x}}{\ln^4 x}}^{0.5\ \text{BASIC SIMPLIFICATION}}$$

$$= \frac{\frac{2\ln x}{x}[1-(1+2\ln(\ln x))]}{\ln^4 x} = \frac{\frac{2\ln x}{x}[-2\ln(\ln x)]}{\ln^4 x} = \frac{-4\ln(\ln x)}{x\ln^3 x}$$

$$= \frac{2\ln x\,[1-(1+2\ln(\ln x))]}{x\ln^4 x}$$

ANY OF THIS 0.5

[5] (c) $f(x) = e^{4x}\sqrt{1-e^{4x}}$

$$f'(x) = \overbrace{\underbrace{4e^{4x}}_{1}\sqrt{1-4e^{4x}} + e^{4x}\cdot\frac{1}{2\sqrt{1-4e^{4x}}}\cdot\underbrace{(-16e^{4x})}_{2}}^{\text{PROD. RULE } 1 + \text{DER. } 3} = \overbrace{4e^{4x}\sqrt{1-4e^{4x}} + \frac{-8e^{8x}}{\sqrt{1-4e^{4x}}}}^{0.5\ \text{BASIC SIMPLIFICATION}}$$

$$= \frac{4e^{4x}(1-4e^{4x}) - 8e^{8x}}{\sqrt{1-4e^{4x}}} = \frac{4e^{4x}[(1-4e^{4x}) - 2e^{4x}]}{\sqrt{1-4e^{4x}}} = \frac{4e^{4x}(1-6e^{4x})}{\sqrt{1-4e^{4x}}}$$

or $4e^{4x}\left[\sqrt{1-4e^{4x}} - \dfrac{2e^{4x}}{\sqrt{1-4e^{4x}}}\right]$

$$= 4e^{4x}\left[\frac{1-4e^{4x}-2e^{4x}}{\sqrt{1-4e^{4x}}}\right] = 4e^{4x}\left[\frac{1-6e^{4x}}{\sqrt{1-4e^{4x}}}\right]$$

ANY OF THIS 0.5

[5] (d) $f(x) = (\sec 4x)^{\frac{1}{x}}$

$$\underbrace{\ln f(x) = \frac{1}{x}\ln(\sec 4x)}_{0.5} = \frac{\ln(\sec 4x)}{x}\ \Big\}\ 1$$

$$\overbrace{\frac{1}{f(x)}\cdot f'(x)}^{0.5} = \overbrace{\frac{x\cdot\frac{1}{\sec 4x}\cdot\sec 4x\tan 4x\cdot 4 - \ln(\sec 4x)\cdot 1}{x^2}}^{\text{QUOT. } 0.5 + \text{DER } 1.5} = \overbrace{\frac{4x\tan 4x - \ln(\sec 4x)}{x^2}}^{0.5}$$

OK IF HERE OR AT END

$$\underbrace{f'(x) = f(x)}_{0.5}\left[\frac{4x\tan 4x - \ln(\sec 4x)}{x^2}\right] = \underbrace{(\sec 4x)^{\frac{1}{x}}}_{0.5}\left[\frac{4x\tan 4x - \ln(\sec 4x)}{x^2}\right]$$

6. Find and simplify the derivative of each of the following: Ⓐ $\frac{d}{dx} \text{ArcSin}\, x = \frac{1}{\sqrt{1-x^2}}$

[4] (a) $f(x) = \sin^{-1}(\sqrt{1-e^{4x}})$ [Note that $\sin^{-1} x = \arcsin x$]

KNOW Ⓐ ① + CHAIN ① + $(e^{4x})'$ ① BASIC SIMPLIFYING 0.5

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{1-e^{4x}})^2}} \cdot \frac{1}{2\sqrt{1-e^{4x}}} \cdot (-4e^{4x}) = \frac{1}{\sqrt{e^{4x}}} \cdot \frac{-2e^{4x}}{\sqrt{1-e^{4x}}} = \frac{-2e^{4x}}{\sqrt{e^{4x}}\sqrt{1-e^{4x}}}$$

$$= \frac{1}{e^{2x}} \cdot \frac{-2e^{4x}}{\sqrt{1-e^{4x}}} = \frac{-2e^{2x}}{\sqrt{1-e^{4x}}}$$ } ANY OF THIS 0.5

[5] (b) $f(x) = \dfrac{\sinh 2x}{1+\cosh 2x}$

QUOT. RULE ① + DER. ③ BASIC SIMPLIFICATION 0.5

$$f'(x) = \frac{(1+\cosh 2x)\cdot 2\cosh 2x - \sinh 2x \cdot 2\sinh 2x}{(1+\cosh 2x)^2} = \frac{2\cosh 2x + 2(\cosh^2 2x - \sinh^2 2x)}{(1+\cosh 2x)^2}$$

$$= \frac{\cosh 2x + 2}{(1+\cosh 2x)^2} = \frac{2(\cosh 2x + 1)}{(1+\cosh 2x)^2} = \frac{2}{1+\cosh 2x}$$ } ANY OF THIS 0.5

7. Find each of the following limits: (BY ANY METHOD INCLUDING BY L'HÔP)

[5] (a) $\lim_{x\to 0} \dfrac{x\sin 2x}{1-\cos 2x}$ $\frac{0}{0}$ ∴ INDET. SO L'HÔP. 0.5

0/0 SO L'HÔP AGAIN 0.5   1.5   1.5   ①

$$\lim_{x\to 0} \frac{x\sin 2x}{1-\cos 2x} = \lim_{x\to 0} \frac{2x\cos 2x + \sin 2x}{2\sin 2x} = \lim_{x\to 0} \frac{-4x\sin 2x + 2\cos 2x + 2\cos 2x}{4\cos 2x} = \frac{0+2+2}{4} = 1$$

$$= \lim_{x\to 0} \frac{-4x\sin 2x + 4\cos 2x}{4\cos 4x} = \frac{0+4}{4} = 1$$

OK IF NOT SIMPLIFIED LIKE THIS

[5] (b) $\lim_{x\to 0}(1+\tan 4x)^{\frac{1}{2x}}$

0.5 0.5 (OK IF NOT STATED EXPLICITLY)

$\lim_{x\to 0}(1+\tan 4x)^{\frac{1}{2x}}$ (1^∞ type) ①   Let $y = \lim_{x\to 0}(1+\tan 4x)^{\frac{1}{2x}}$ 1.5

$$\ln y = \lim_{x\to 0} \frac{1}{2x}\ln(1+\tan 4x) = \lim_{x\to 0} \frac{\ln(1+\tan 4x)}{2x} = \lim_{x\to 0} \frac{\frac{1}{1+\tan 4x}\cdot 4\sec^2 4x}{2}$$

$\frac{\ln 1}{0} = \frac{0}{0}$ ∴ L'HÔP. 0.5

$$= \lim_{x\to 0} \frac{2\sec^2 4x}{1+\tan 4x} = \frac{2}{1+0} = 2$$ 0.5

So $y = \lim_{x\to 0}(1+\tan 4x)^{\frac{1}{2x}} = e^2$ ①

[6] 8. Find an equation of the normal line to the graph of the equation

$$2x^3 - 3x^2y = 2y^2 - 3$$

at the point $(2, 3)$.

(i) y' IMPLICITLY

$$2x^3 - 3x^2y = 2y^2 - 3$$

$$6x^2 - 3x^2y' - 6xy = 4yy' \quad (1.5)$$

$$(4y + 3x^2)y' = 6x^2 - 6xy \quad (1)$$

$$y' = \frac{6x(x-y)}{4y + 3x^2} \quad (1)$$

(3.5)

(ii) $y'\big|_{(2,3)}$

At $(2,3)$: $y' = \dfrac{12(2-3)}{4(3) + 3(2)^2} = \dfrac{-12}{24} = -\dfrac{1}{2}$ (1)

(iii)

$\underbrace{m_N = -\dfrac{1}{y'} = 2}_{0.5}$ Equation: $y - 3 = 2(x - 2)$

$\underbrace{y = 2x - 1}_{1}$

(1.5)

6

[5] 9. A piece of ice in the shape of a right circular cylinder is melting. If its radius is decreasing at a rate of 2 cm/hr and its height is decreasing at a rate of 5 cm/hr, at what rate is the volume changing when the volume is 90π cm^3 and the radius is 3 cm?

(1) RATE GIVEN $\dfrac{dr}{dt} = -2$, $\dfrac{dh}{dt} = -5$

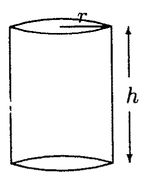
(1.5) (2) RATE WANTED dV/dt

(3) WHEN: $V = 90$, $r = 3$ (1.5) (7) IMPLICIT DIFF'N

(4) ID VARIABLES: r, h, V (1) (8) PLUG-IN

(1) (5) REL'N: $V = \pi r^2 h$

(6) 3 VAR. BUT OK AS 2 RATES GIVEN



$V = \pi r^2 h$

STEP #1 STEP #2 STEP #3

Given $\underbrace{\dfrac{dr}{dt} = -2 \text{ and } \dfrac{dh}{dt} = -5}_{0.5}$, find $\underbrace{\dfrac{dV}{dt}}_{0.5}$ $\underbrace{\text{when } V = 90\pi \text{ and } r = 3.}_{0.5}$ (1.5)

STEP #4 r, h, V STEP #5 $V = \pi r^2 h$ STEP #6 OK STEP #7

(1) $\dfrac{dV}{dt} = \pi\left[r^2\dfrac{dh}{dt} + 2rh\dfrac{dr}{dt}\right] = \pi(-5r^2 - 4rh)$ (1.5) (2.5)

STEP #8

$V = 9\pi$ and $r = 3 \Rightarrow 90\pi = \pi \cdot 3^2 h \Rightarrow h = 10$. Then

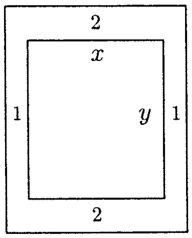
$$\frac{dV}{dt} = \pi[-5(3)^2 - 4(3)(10) = \pi(-45 - 120) = -165\pi \text{ cm}^3/\text{hr}$$

(1)

5

OK IF DIFFERENT STEPS USED

[8] 10. A rectangular page is to contain 32 square inches of print, surrounded by 2 inch margins at the top and bottom of the page and 1 inch margins on each side. Find the dimensions of the page so that the least amount of paper is used.



Let x = width of printed part of the page
y = length of printed part of the page
A = area of page

2.5 We want to minimize

$$A = (x+2)(y+4) \qquad \text{with} \quad xy = 32$$
$$y = \frac{32}{x}$$

OR $A = x\,y$ $\qquad (x-2)(y-4) = 32 \Rightarrow y = \frac{32}{x-2} + 4$

1 $$A(x) = (x+2)\left(\frac{32}{x}+4\right) = 32 + 4x + \frac{64}{x} + 8 = 4x + \frac{64}{x} + 40\ , \quad 0 < x < \infty$$

2 $$A'(x) = 4 - \frac{64}{x^2} = \frac{4x^2-64}{x^2} = \frac{4(x^2-16)}{x^2}$$
$$A'(x) = 0 \quad \text{when} \quad x = 4 \qquad [-4 \notin D_V]$$

1.5 $$A''(x) = \frac{128}{x^3} \Rightarrow A''(4) = \frac{128}{4^3} > 0.$$ So A has a minimum at $x = \cancel{20}\ 4$. OR 1st Der. Test

1 Then $y = \frac{32}{x} = \frac{32}{4} = 8$. So the dimensions of the page should be

$(4+2)$ by $(8+4)$ i.e. 6 in by 12 in

8

0.5 Step #1 What's being optimized — Least amount of paper — Minimize area

1 Step #2 Rel'n for #1 — If x, y printed area: $A = (x+2)(y+4)$ OR If x, y overall area: $A = x\,y$

2 Step #3 3 var. so another rel'n — $xy = 32$, $y = \frac{32}{x}$, $y = 32x^{-1}$ OR $(x-2)(y-4) = 32$, $y - 4 = \frac{32}{x-2}$, $y = \frac{32}{x-2} + 4$, $y = 32(x-2)^{-1} + 4$

2 Step #4 Der. + CPs

$A = (x+2)(32x^{-1}+4)$
$A' = (1)(32x^{-1}+4) + (x+2)(32(-1)x^{-2})$
$= \frac{32}{x} + 4 + (x+2)\left(-\frac{32}{x^2}\right)$
$= \frac{32x + 4x^2 - 32(x+2)}{x^2}$
$= \frac{\cancel{32x} + 4x^2 - \cancel{32x} - 64}{x^2}$
$= \frac{4x^2-64}{x^2}$
$= \frac{4(x^2-16)}{x^2}$

CPs: $x^2 - 16 = 0$, $x = 4$, $\cancel{-4}$ (Neg. width)
$x^2 = 0$, $\cancel{x=0}$ (No printing)
$\boxed{x=4}$

$A = x\left[32(x-2)^{-1}+4\right]$
$A' = (1)\left[32(x-2)^{-1}+4\right] + x\left[32(-1)(x-2)^{-2}\right]$
$A' = \frac{32}{x-2} + 4 - \frac{32x}{(x-2)^2}$
$= \frac{32(x-2) + 4(x-2)^2 - 32x}{(x-2)^2}$
$= \frac{32x - 64 + 4x^2 - 16x + 16 - 32x}{(x-2)^2}$
$= \frac{4x^2 - 16x - 48}{(x-2)^2}$
$= \frac{4(x^2-4x-12)}{(x-2)^2}$
$= \frac{4(x-6)(x+2)}{(x-2)^2}$

CPs: $x = 6$, $\cancel{-2}$, $\cancel{2}$
$\boxed{x=6}$

1.5 Step #5 Rel max/mins

$x = 4$

	$x<4$	$x>4$
TP	3	5
A'	$-/+ = \ominus$	$+/+ = \oplus$
A	↓	↑

∴ Rel. min = Abs. min at $x = 4$

$x = 4$ so $y = \frac{32}{4} = 8$

Printed area 4×8
Paper 6×12

OR

$x = 6$

	$x<6$	$x>6$
TP	5	7
A'	$-/+ = \ominus$	$+/+ = \oplus$
A	↓	↑

∴ Rel. min = Abs. min at $x = 6$

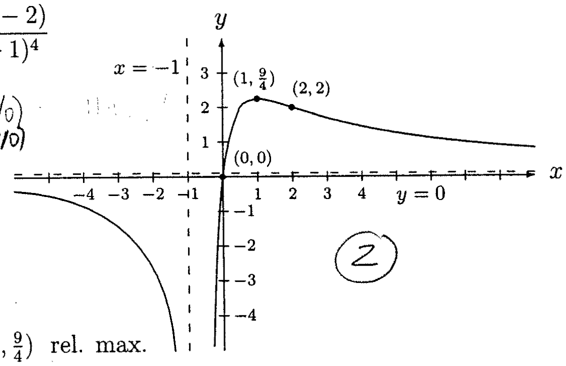
$x = 6$ so $y = 12$

Paper 6×12

Step #6
1

[10] 11. Sketch the graph of $y = \dfrac{9x}{(x+1)^2}$, giving intercepts, asymptotes, where increasing and where decreasing, any relative maximum and relative minimum points, where concave upward, where concave downward, and any inflection points. $\left[\text{Note: } y' = \dfrac{9(1-x)}{(x+1)^3}\right.$ and $\left. y'' = \dfrac{18(x-2)}{(x+1)^4}\right]$

$y = \dfrac{9x}{(x+1)^2}$, $y' = \dfrac{9(1-x)}{(x+1)^3}$, $y'' = \dfrac{18(x-2)}{(x+1)^4}$



1 Intercepts: (0, 0)

2 Asymptotes: $y = 0$, $x = -1$ TEST: VA (a/0) NOT HOLE (0/0)

TEST: CROSSED (0,0)

Other:

1 CPs $\begin{cases} y' = 0 \text{ for } x = 1 & \Rightarrow 9(1-x) = 0 \therefore x = 1 \\ y' = \infty \text{ for } x = -1 & \Rightarrow x+1 = 0 \therefore x = -1 \end{cases}$

1.5 TEST CPs REL. EXT.

$-\infty < x < -1$	$y' < 0$	decreasing	
$-1 < x < 1$	$y' > 0$	increasing	$(1, \frac{9}{4})$ rel. max.
$1 < x < \infty$	$y' < 0$	decreasing	

1 FPs $\begin{cases} y'' = 0 \text{ for } x = 2 & 18(x-2) = 0 \Rightarrow x = 2 \\ y'' = \infty \text{ for } x = -1 & x+1 = 0 \Rightarrow x = -1 \end{cases}$

1.5 TEST FPs CONC.

$-\infty < x < -1$	$y'' < 0$	concave down	
$-1 < x < 2$	$y'' < 0$	concave down	$(2, 2)$ inflection point
$2 < x < \infty$	$y'' > 0$	concave up	

10

A) f(x)

1 ① X- & Y- INT.

2 ② VA + TEST, HA + TEST, NOTE (−0.5) IF NO TESTS

B) f′(x)

1 ① CPs

1.5 ② ↑ ↓ ③ REL. MAX. / MIN

C) f″(x)

1 ① FPs

1.5 ② ∩ ∪ ③ ACTUAL FPs

2 D) SKETCH
① SUMMARIZE A–C WITH Y-VALUES
② PLOT INTERCEPTS, VA/HA, REL MAX. / MIN., FPs
③ JOIN WITH CONCAVITY

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OK IF DIFFERENT STEPS

[5] 12. Answer **one** of the following:

(a) If $f(x)$ is nonzero, differentiable and increasing on the open interval (a, b), show that the function $g(x) = \dfrac{1}{f(x)}$ is decreasing on (a, b).

(b) Given the equation $x^4 + y^4 = a^4$, where a is a constant, show that

$$y'' = -\frac{3a^4x^2}{y^7}$$

(a) Since $f(x)$ is increasing on (a, b), $f'(x) > 0$ on (a, b), and then, since $f(x) \neq 0$ on (a, b) it follows that

$$g'(x) = -\frac{f'(x)}{[f(x)]^2} < 0$$

on (a, b) since $[f(x)]^2 > 0$. So $g(x)$ is decreasing on (a, b).

(b) $x^4 + y^4 = a^4$

$\overline{100}$ 2 $4x^3 + 4y^3y' = 0$ 1

$y' = -\dfrac{4x^3}{4y^3} = -\dfrac{x^3}{y^3}$ 1

1 0.5

3 $y'' = -\dfrac{y^3 \cdot 3x^2 - x^3 \cdot 3y^2y'}{(y^3)^2} = \dfrac{-3x^2y^3 + 3x^3y^2 \cdot \left(-\frac{x^3}{y^3}\right)}{y^6} = \dfrac{-3x^2y^3 - \frac{3x^6}{y}}{y^6}$

$= \dfrac{-3x^2y^4 - 3x^6}{y^7} = \dfrac{-3x^2(y^4 + x^4)}{y^7} = -\dfrac{3x^2a^4}{y^7}$

$y^4 + x^4 = a^4$ 0.5

1

5

a) $g(x) = \dfrac{1}{f(x)} = [f(x)]^{-1}$

1.5 $g'(x) = (-1)[f(x)]^{-2} f'(x)$ By CHAIN RULE

$g'(x) = \dfrac{-1}{[f(x)]^2} f'(x)$

1.5 GIVEN: $f(x) \uparrow$, $f'(x) > 0$ on (a,b)

$f(x) \neq 0$

REALIZE: $[f(x)]^2 > 0$ As $[\ ]^2$ & $f(x) \neq 0$

2 $\therefore g'(x) = \dfrac{-1}{\oplus} (\oplus) = \ominus$

$\therefore g(x) \downarrow$ ON (a,b)

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