MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION | **Mathematics 1000** | WINTER 2015

COMPLETE THE FOLLOWING CAREFULLY AND CLEARLY:

(Please Print)

Surname: Solutions

Given Names: ______________________

MUN Number: ______________________

Circle the Name of Your Instructor Below:

J. Craighead   C-Z. Kuo   C. Leonard

	MARKS
12	1. ______
6	2. ______
7	3. ______
17	4. ______
6	5. ______
7	6. ______
6	7. ______
9	8. ______
17	9. ______
8	10. ______
5	11. ______
100	Total ______

Please note:

This exam has **TWELVE** pages, including this one.

The questions are to be answered in the spaces provided.

Under no circumstances may the candidate take this book from the examination room.

On no account are pages to be torn or removed from this book, unless specifically directed.

Candidates must not have in their possession books, notes or papers of any kind, unless specifically directed.

No electronic devices of any kind, including cell phones and MP3 players, are permitted at your desk. Calculators are **NOT** permitted.

FOR INSTRUCTOR'S USE ONLY

FINAL 55%	TERM 45%	TOTAL 100%	FINAL MARK	GRADE

Marks

1. Using methods learned in this course, evaluate the following limits. Show your work. Do not use l'Hôpital's rule

[3] a) $\lim_{x\to -2} \dfrac{x^2+5x+6}{x^2-2x-8}$

$= \lim_{x\to -2} \dfrac{(x+2)(x+3)}{(x+2)(x-4)}$

$= \lim_{x\to -3} \dfrac{x+3}{x-4}$

$= \dfrac{-2+3}{-2-4} = -1/6$

[3] b) $\lim_{x\to 3} \dfrac{4-\sqrt{x^2+7}}{x^2-9}$

$= \lim_{x\to 3} \left(\dfrac{4-\sqrt{x^2+7}}{x^2-9}\right)\left(\dfrac{4+\sqrt{x^2+7}}{4+\sqrt{x^2+7}}\right)$

$= \lim_{x\to 3} \dfrac{16-(x^2+7)}{(x^2-9)(4+\sqrt{x^2+7})}$

$= \lim_{x\to 3} \dfrac{9-x^2}{(x^2-9)(4+\sqrt{x^2+7})}$

$= \lim_{x\to 3} \dfrac{-1}{4+\sqrt{x^2+7}} = \dfrac{-1}{4+\sqrt{16}} = \dfrac{-1}{8}$

[3] c) $\lim_{x\to 7^-} \dfrac{|x^2-49|}{x-7}$ as $x \to 7^-$ $x^2-49<0$ $|x^2-49| = (-1)(x^2-49)$

$= \lim_{x\to 7^-} \dfrac{(-1)(x^2-49)}{x-7}$

$= \lim_{x\to 7^-} \dfrac{(-1)(x+7)(x-7)}{x-7}$

$= \lim_{x\to 7^+} (-1)(x+7) = (-1)(7+7) = -14$

[3] d) $\lim_{x \to 0} \frac{\sin^2(4x)}{3x^2 \cos^2(4x)}$

$= \frac{1}{3} \lim_{x \to 0} \frac{(\sin 4x)(\sin 4x)}{(x)\quad(x)} \left(\frac{1}{\cos^2 4x}\right)$

$= \frac{16}{3} \lim_{x \to 0} \left(\frac{\sin 4x}{4x}\right)\left(\frac{\sin 4x}{4x}\right)\left(\frac{1}{\cos^2 4x}\right)$

$= \left(\frac{16}{3}\right)(1)(1)(1) = 16/3$

[6] 2. Let $f(x) = \begin{cases} \sqrt{4+x} & x < 0 \\ 1 & x = 0 \\ \frac{e^x + 1}{1-x} & x > 0 \end{cases}$

a) Use the Definition of Continuity to determine if $f(x)$ is continuous at $x = 0$. Classify any discontinuities as removable or non-removable.

$F(0) = 1$ $\quad \lim_{x \to 0^-} F(x) = \lim_{x \to 1^-} \sqrt{4+x} = \sqrt{4} = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^x+1}{1-x} = \frac{2}{1} = 2. \quad \therefore \lim_{x \to 0} f(x) = 2$

$\lim_{x \to 0} f(x) \neq f(0)$

$f(x)$ is not Continuous at $x=0$

Removable discontinuity

b) Find any removable or non-removable discontinuities, if they exist.

$1 - x = 0$ when $x = 1$ $\quad f(1) = \frac{e+1}{0}$ undefined

Non removable discontinuity

[7] 3. Use the DEFINITION OF DERIVATIVE to find $f'(x)$ for $f(x)=\sqrt{2x-1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2x+2h-1} - \sqrt{2x-1}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{2x+2h-1} - \sqrt{2x-1}}{h}\right)\left(\frac{\sqrt{2x+2h-1} + \sqrt{2x-1}}{\sqrt{2x+2h-1} + \sqrt{2x-1}}\right)$$

$$= \lim_{h\to 0} \frac{(2x+2h-1) - (2x-1)}{(h)(\sqrt{2x+2h-1} + \sqrt{2x-1})}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{(\cancel{h})(\sqrt{2x+2h-1} + \sqrt{2x-1})}$$

$$= \frac{2}{\sqrt{2x-1} + \sqrt{2x-1}}$$

$$= \frac{2}{2\sqrt{2x-1}} = \frac{1}{\sqrt{2x-1}}$$

4. Differentiate each function and make any appropriate simplifications.

[4] (a) $y = \tan^3\left(\sqrt{3x^3 - x^2}\right)$ DO NOT USE LOGARITHMIC DIFFERENTIATION.

$$y' = 3\tan^2\left(\sqrt{3x^3-x^2}\right)\left(\sec^2\left(\sqrt{3x^3-x^2}\right)\right)\left(\frac{1}{2}\left(3x^3-x^2\right)^{-\frac{1}{2}}\left(9x^2-2x\right)\right)$$

$$= \left(\frac{3}{2}\right)\left[\frac{(9x^2-2x)\left[\tan^2\left(\sqrt{3x^3-x^2}\right)\right]\left[\sec^2\left(\sqrt{3x^3-x^2}\right)\right]}{\sqrt{3x^3-x^2}}\right]$$

[4] (b) $y = \dfrac{2x^2 - 3x}{(4x-3)^4}$ DO NOT USE LOGARITHMIC DIFFERENTIATION.

$$y' = \frac{(4x-3)(4x-3)^4 - (2x^2-3x)\left(4(4x-3)^3\right)(4)}{\left((4x-3)^4\right)^2}$$

$$= \frac{(4x-3)^3\left[(4x-3)^2 - 16(2x^2-3x)\right]}{(4x-3)^8}$$

$$= \frac{16x^2 - 24x + 9 \quad -32x^2 + 48x}{(4x-3)^5}$$

$$= \frac{-16x^2 + 24x + 9}{(4x-3)^5}$$

[4] (c) Use logarithmic to find y' for $y = \dfrac{e^{6x}}{\left(\sqrt{3x^2-2}\right)\left(3+5x^2\right)^4}$

$$\ln y = \ln(e^{6x}) - \ln(3x^2-2)^{\frac{1}{2}} - \ln(3+5x^2)^4$$

$$\ln y = 6x - \tfrac{1}{2}\ln(3x^2-2) - 4\ln(3+5x^2)$$

$$\left(\tfrac{1}{y}\right)y' = 6 - \frac{(6x)}{(2)(3x^2-2)} - \frac{4(10x)}{3+5x^2}$$

$$y' = \frac{e^{6x}}{\left(\sqrt{3x^2-2}\right)(3+5x^2)^4}\left[6 - \frac{3x}{3x^2-2} \quad \frac{-40x}{3+5x^2}\right]$$

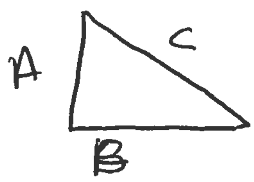
[5] (d) $y = (\cos x)^{(\sinh x)}$

$$\ln y = [\sinh(x)]\,[\ln(\cos x)]$$

$$\left(\tfrac{1}{y}\right)y' = [\cosh(x)]\,[\ln(\cos x)] + [\sinh(x)]\left[-\frac{\sin x}{\cos x}\right]$$

$$y' = (\cos x)^{(\sinh x)}\left([\cosh(x)]\,[\ln(\cos x)] - [\sinh(x)]\,[\tan x]\right)$$

[6] 5. Ship A leaves point P at the 10 AM and sail North at 15 kilometers an hour. Ship B leaves point P at 11 AM and sails west at 12 kilometers an hour. How fast is the distance between them changing at noon ?



$\frac{dA}{dt} = 15$ km/hours $\quad \frac{dB}{dt} = 12$ km/hr

Need $\frac{dC}{dt}$ when $A = 30 \quad B = 12$

$\sqrt{30^2+12^2} = \sqrt{900+144} = \sqrt{1044} = 6\sqrt{29}$

$A^2 + B^2 = C^2$

$\cancel{2}A\frac{dA}{dt} + \cancel{2}B\frac{dB}{dt} = \left(\cancel{2}\frac{dC}{dt}\right)(C)$

$(30)(15) + (12)(12) = (6\sqrt{29})\left(\frac{dC}{dt}\right)$

$\frac{450+144}{6\sqrt{29}} = \frac{dC}{dt}$

$\frac{594}{6\sqrt{29}} = \frac{dC}{dt} = \frac{99}{\sqrt{29}} = \frac{99\sqrt{29}}{29}$ km/hour

[7] 6.A closed box is to be manufactured with its length 3 times its width and with a volume of 36 cubic meters. What dimensions will require the minimum material for construction?

$SA = (2L)(3L) + (2)(L)(h) + (2)(3L)(h)$

$= 6L^2 + 8Lh$

$V = 3L^2h = 36 \quad h = \frac{36}{\cancel{3}L^2} = \frac{12}{L^2}$

$SA = 6L^2 + (8L)\left(\frac{12}{L^2}\right) = 6L^2 + 96L^{-1}$

$SA' = 12L - 96L^{-2} \qquad SA'' = 12 + 192L^{-3}$

if $L > 0 \quad SA'' > 0$

$SA' = 0$ when $\quad 12L - \frac{96}{L^2} = 0 \qquad 12L^3 - 96 = 0$

$L^3 = 8 \quad L = 2$

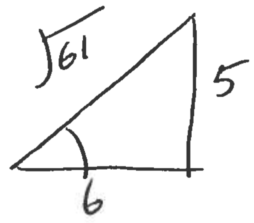
$SA'' > 0$ if $L = 0$ or $h = 0$ do not have a Box

$L = 2$

$3L = 6$

$h = \frac{12}{4} = 3$

[3] 7. a) Simplify the expression $\sin\left(\arctan\left(\frac{5}{6}\right)\right)$



$$\sin\left(\arctan\left(\tfrac{5}{6}\right)\right) = \frac{5}{\sqrt{61}} = \frac{5\sqrt{61}}{61}$$

[3] b) If $f(x) = \arctan(3x^2)$ find $f'(x)$

$$f'(x) = \frac{1}{1+(3x^2)^2}(3x^2)'$$

$$= \frac{6x}{1+9x^4}$$

[9] 8 a) Use implicit differentiation to find y' for $x^2 + 3xy^2 + y^3 = 9 + xy$

[3] $2x + 3y^2 + 6xyy' + 3y^2y' = y + xy'$

$6xyy' + 3y^2y' - xy' = y - 2x - 3y^2$

$$y' = \frac{y - 2x - 3y^2}{6xy + 3y^2 - x}$$

b) Find the equation of the tangent line at (2,1)

[2] $y' = \dfrac{1 - 2(2) - 3(1)^2}{(6)(2)(1) + 3(1)^2 - 2}$

$= \dfrac{-6}{13}$

$\dfrac{y-1}{x-2} = \dfrac{-6}{13}$

$13y - 13 = -6x + 12$

$13y = -6x + 25$

$y = \dfrac{-6x+25}{13}$

c) If $y = e^{(xy^2)}$ find y'

[4] $y' = e^{(xy^2)}\left[y^2 + 2xyy'\right]$

$y' = (y^2)(e^{(xy^2)}) + (2xyy')(e^{(xy^2)})$

$y' - 2xyy'(e^{(xy^2)}) = y^2(e^{(xy^2)})$

$$y' = \frac{(y^2)(e^{(xy^2)})}{1 - 2xy(e^{(xy^2)})}$$

9. Given the following

$$f(x)=\frac{x-1}{(x-2)^2} \qquad f'(x)=\frac{-x}{(x-2)^3} \qquad f''(x)=\frac{2x+2}{(x-2)^4}$$

3 [3] a) Find the vertical asymptotes of $f(x)$, if any. Justify your answer using Limits.

$(x-2)^2 = 0$ when $x=2$

$$\lim_{x\to 2^-} \frac{x-1}{(x-2)^2} = \frac{1}{(0^-)^2} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to 2^+} \frac{x-1}{(x-2)^2} = \frac{1}{(0^+)^2} = \frac{1}{0^+} = +\infty$$

$x=2$ is a vertical asymptote

3 [3] b) Find the horizontal asymptotes of $f(x)$, if any. Justify your answer using Limits

$$F(x) = \frac{x-1}{x^2-4x+4}$$

$$\lim_{x\to\pm\infty} \frac{x-1}{x^2-4x+4} = \lim_{x\to\pm\infty} \frac{\frac{x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{4}{x^2}} = \lim_{x\to\pm\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}} = \frac{0}{1} = 0$$

$y=0$ Horizontal asymptote

[2] c) Find the x and y intercepts of the graph $f(x)$, if any.

y intercept $x=0$ $\quad \frac{-1}{(-2)^2} = \frac{-1}{4} = -\frac{1}{4}$

x intercept $\quad \frac{x-1}{(x-2)^2} = 0 \qquad x-1=0 \quad x=1$

3 [3] d)Determine the intervals on which $f(x)$ is increasing or decreasing and classify any relative (local) extrema.

$$f'(x) = \frac{-x}{(x-2)^3}$$

$f'(x)$ DNE at $x=2$

$f'(x)=0$ when $-x=0$ $\quad x=0$

		0		2	
$f'(x)$	<0	$=0$	>0	DNE	<0

$f(x)$ increasing on $(0,2)$

decreasing on $(-\infty,0) \cup (2,\infty)$

3[2] e) Determine the intervals on which $f(x)$ is concave up or concave down and identify any inflection points.

$f''(x) = \frac{2x+2}{(x-2)^4}$

$f''(x)$ DNE at $x=2$

$f''(x) = 0$ when $2x+2=0$ $\quad x=-1$

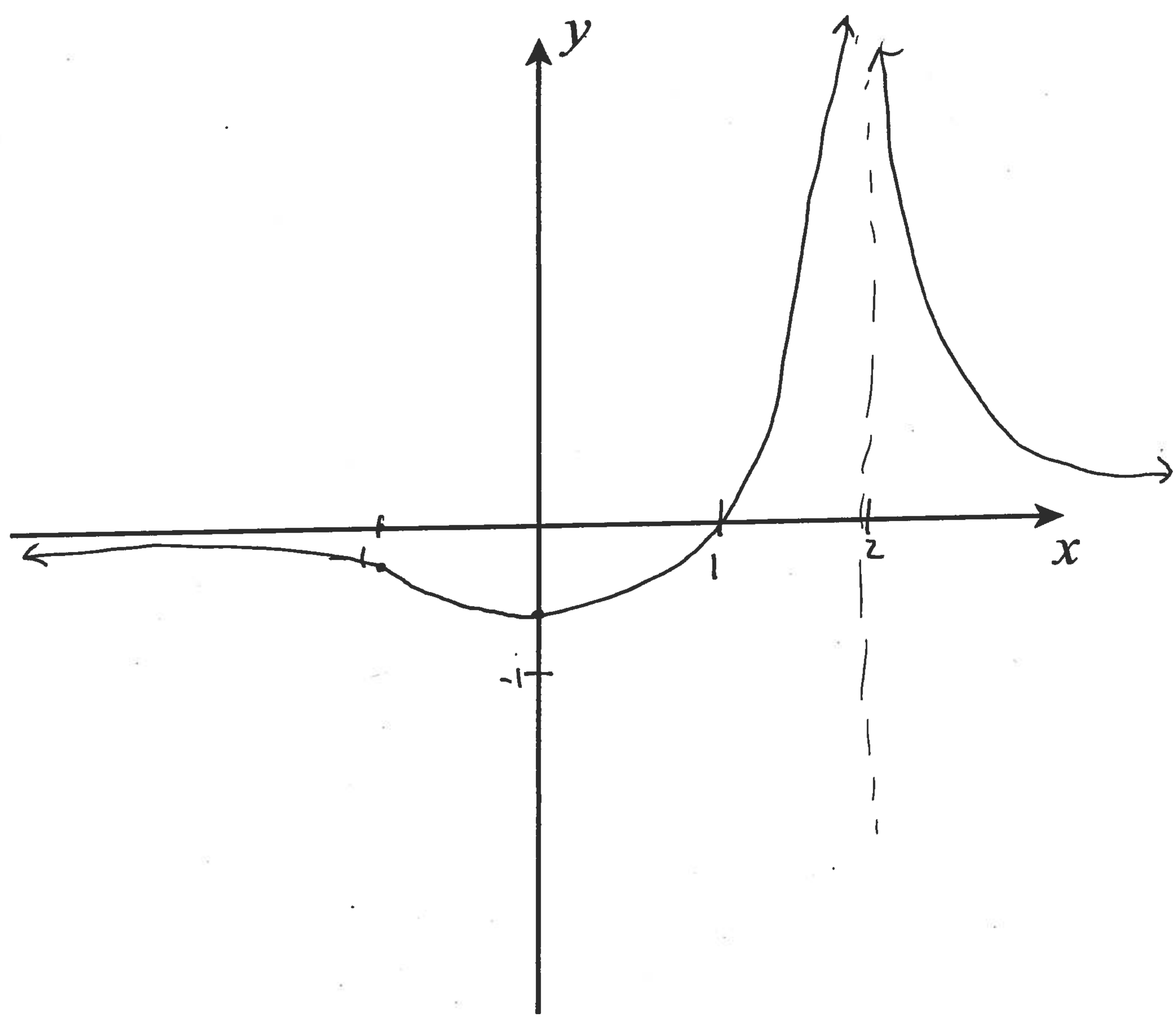
		-1		2	
$f''(x)$	<0	$= 0$	>0	DNE	>0

$f(x)$ Concave up on $(-1, 2) \cup (2, \infty)$

Concave down on $(-\infty, -1)$

$f(-1) = \frac{-2}{(-1-2)^2} = \frac{-2}{9}$

[3] f) Sketch the graph of $f(x)$ on the axes provided. Label any inflection points and extrema.



10. Use l'Hôpital's rule to find the following limits

[4] a) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{1/x}$$

$$= \lim_{x\to\infty} \frac{\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$$

[4] b) $\lim_{x\to 0^+} (1+4x)^{\frac{3}{x}}$

$$\ln(f(x)) = \frac{3}{4}\ln(1+4x)$$

$$\lim_{x\to 0^+} \ln(f(x)) = \lim_{x\to 0^+} \frac{3}{4}\left(\ln(1+4x)\right)$$

$$= \lim_{x\to 0^+} \left(\frac{3}{4}\right)\left(\frac{4}{1+4x}\right) = \lim_{x\to 0^+} \frac{3}{1+4x} = 3$$

$$\lim_{x\to 0^+} (1+4x)^{3/4} = \lim_{x\to 0^+} e^{\lim_{x\to 0^+} \ln(f(x))} = e^3$$

[5] 11. Answer **ONE** of the following:

a) Given $y = ax^3 + bx^2 + cx + d$, find a, b, c, d such that the y intercept = 3, critical numbers at $x = 3$ *and* $x = 2$ and an inflection point at $x = \frac{5}{2}$

or

b) prove that if $y = \arcsin(x)$ Then $y' = \frac{1}{\sqrt{1-x^2}}$

a) $y' = 3ax^2 + 2bx + c$ with root $x=3$, $x=2$

$d = 3$

$(x-3)(x-2) = x^2 - 5x + 6 = 0$

LCD $(3,2) = 6$

$6x^2 - 30x + 36 = 0$

$3a = 6 \quad a = 2 \qquad 2b = -30 \quad b = -15, \qquad c = 36$

$2x^3 - 15x^2 + 36x + 3$

$y'' = 12x - 30 = 0$ when $x = \frac{30}{12} = \frac{5}{2}$

Note: multiples of 2, −15 and 36 eg $(4, -30, 72)$ also work.

11b $y = \arcsin x$

$\sin y = \sin(\arcsin x)$

$\sin y = x$

$(\cos y) y' = 1$

$y' = \frac{1}{\cos y}$

$y' = \frac{1}{\sqrt{1-x^2}}$