MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION | **Mathematics 1000** | WINTER 2014

COMPLETE THE FOLLOWING CAREFULLY AND CLEARLY:

(Please Print)

Surname: Solutions

Given Names: ______________________

MUN Number: ______________________

Instructor: Austin Leonard Suvak Wang

	MARKS
9	1.
4	2.
10	3.
5	4.
20	5.
10	6.
9	7.
5	8.
5	9.
8	10.
10	11.
5	12.
100	Total

Please note:

This exam has **EIGHT** pages of questions.

All calculators are strictly forbidden.

The questions are to be answered in the spaces provided.

Under no circumstances may the candidate take this book from the examination room.

On no account are pages to be torn or removed from this book, unless specifically directed.

Candidates must not have in their possession books, notes or papers of any kind, unless specifically directed.

No electronic devices of any kind, including cell phones and MP3 players, are permitted at your desk.

FOR INSTRUCTOR'S USE ONLY

FINAL 55%	TERM 45%	TOTAL 100%	FINAL MARK	GRADE

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL EXAMINATION MATHEMATICS 1000 Winter, 2014

Calculators are not permitted on this examination.

1. Evaluate each of the following limits, assigning ∞ or $-\infty$ where appropriate. You may not use L' Hospital's Rule.

[3] (a) $\lim\limits_{x\to 2} \dfrac{3-\sqrt{x^2+5}}{x-2}$

$$\lim_{x\to 2} \frac{3-\sqrt{x^2+5}}{x-2} = \lim_{x\to 2} \frac{3-\sqrt{x^2+5}}{x-2} \cdot \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}} = \lim_{x\to 2} \frac{9-(x^2+5)}{(x-2)(3+\sqrt{x^2+5})}$$

$$= \lim_{x\to 2} \frac{4-x^2}{(x-2)(3+\sqrt{x^2+5})} = \lim_{x\to 2} \frac{(2-x)(2+x)}{(x-2)(3+\sqrt{x^2+5})} = \lim_{x\to 2} \frac{-(2+x)}{3+\sqrt{x^2+5}} = \frac{-4}{6} = -\frac{2}{3}$$

[3] (b) $\lim\limits_{x\to\infty} \dfrac{8x-5}{\sqrt{8x^3+1}}$

$$\lim_{x\to\infty} \frac{8x-5}{\sqrt[3]{8x^3+1}} = \lim_{x\to\infty} \frac{\frac{1}{x}(8x-5)}{\frac{1}{x}\sqrt[3]{8x^3+1}} = \lim_{x\to\infty} \frac{8-\frac{5}{x}}{\sqrt[3]{8+\frac{1}{x^3}}} = \frac{8-0}{\sqrt[3]{8+0}} = \frac{8}{\sqrt[3]{8}} = \frac{8}{2} = 4$$

[3] (c) $\lim\limits_{x\to 1^+} \dfrac{x-3}{2x^2-5x+3}$

$$\lim_{x\to 1^+} \frac{x-3}{2x^2-5x+3} = \lim_{x\to 1^+} \frac{x-3}{\underbrace{(2x-3)}_{-}\underbrace{(x-1)}_{+}} = \frac{-2}{0^-} = \infty$$

[4] 2. Find the horizontal and vertical asymptotes, if any, for the graph of the function

$$f(x) = \frac{(2x-1)^2}{4x^2-1}$$

(1) $\displaystyle\lim_{x\to\infty} \frac{(2x-1)^2}{4x^2-1} = \lim_{x\to\infty} \frac{4x^2-4x+1}{4x^2-1} = \lim_{x\to\infty} \frac{4-\frac{4}{x}+\frac{1}{x^2}}{4-\dfrac{1}{x^2}} = \frac{4-0+0}{4-0} = 1$

So the line $y=1$ is a horizontal asymptote.

(2) $\displaystyle f(x) = \frac{(2x-1)^2}{4x^2-1} = \frac{(2x-1)^2}{(2x-1)(2x+1)} = \frac{2x-1}{2x+1}$

$2x+1=0$ when $x=-\frac{1}{2}$. So the line $x=-\frac{1}{2}$ is a vertical asymptote.

[4] 3. (a) Determine whether the function

$$f(x) = \begin{cases} \dfrac{x^3 - 64}{x^2 - 16} & \text{if } x \neq 4 \\ 0 & \text{if } x = 4 \end{cases}$$

is continuous at $x = 4$. Justify your answer using the definition of continuity.

$$\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16} = \lim_{x \to 4} \frac{(x-4)(x^2+4x+16)}{(x-4)(x+4)} = \lim_{x \to 4} \frac{x^2+4x+16}{x+4} = \frac{16+16+16}{4+4}$$
$$= \frac{48}{8} = 6$$

But $f(4) = 0$. So $\lim_{x \to 4} f(x) \neq f(4)$, and f is not continuous at $x = 4$.

[6] (b) Use the definition of continuity to find a and b, if possible, so that the function

$$f(x) = \begin{cases} ax + b & \text{if } x < 2 \\ 6 & \text{if } x = 2 \\ ax^2 - b & \text{if } x > 2 \end{cases}$$

is continuous at $x = 2$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (ax + b) = 2a + b$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax^2 - b) = 4a - b$$

For $\lim_{x \to 2} f(x)$ to exist we must have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$
$$2a + b = 4a - b$$
$$a = b$$

Then $\lim_{x \to 2} f(x) = 3a$. Now $f(2) = 6$. So to have $\lim_{x \to 2} f(x) = f(2)$ we must have $3a = 6$; i.e. $a = 2$. So for f to be continuous at $x = 2$ we must have $a = b = 2$.

[5] 4. Use the definition of the derivative to find the derivative of $f(x) = \dfrac{5}{3 - x^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{5}{3-(x+h)^2} - \dfrac{5}{3-x^2}}{h} = \lim_{h \to 0} \frac{5(3-x^2) - 5[3-(x+h)^2]}{[3-(x+h)^2](3-x^2)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{15 - 5x^2 - 15 + 5x^2 + 10xh + 5h^2}{[3-(x+h)^2](3-x^2)} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{10xh + 5h^2}{[3-(x+h)^2](3-x^2)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{10x + 5h}{[3-(x+h)^2](3-x^2)} = \frac{10x}{(3-x^2)(3-x^2)} = \frac{10x}{(3-x^2)^2}$$

5. Find and simplify the derivative of each of the following. Use logarithmic differentiation only where necessary:

[5] (a) $f(x) = \dfrac{\sin^2 2x}{1 + \cos^2 2x}$

$$f'(x) = \frac{(1+\cos^2 2x)\cdot 2\sin 2x \cdot \cos 2x \cdot 2 - \sin^2 2x \cdot 2\cos 2x \cdot (-\sin 2x)\cdot 2}{(1+\cos^2 2x)^2}$$

$$= \frac{4(1+\cos^2 2x)\sin 2x \cos 2x + 4\sin^3 2x \cos 2x}{(1+\cos^2 2x)^2} = \frac{4\sin 2x \cos 2x\,[(1+\cos^2 2x)+\sin^2 2x]}{(1+\cos^2 2x)^2}$$

$$= \frac{4\sin 2x \cos 2x\,(1+1)}{(1+\cos^2 2x)^2} = \frac{8\sin 2x\cos 2x}{(1+\cos^2 2x)^2}$$

[5] (b) $f(x) = \dfrac{e^{4x}}{\sqrt{1 - e^{4x}}}$

$$f'(x) = \frac{\sqrt{e^{4x}-1}\cdot e^{4x}\cdot 4 - e^{4x}\cdot \dfrac{1}{2\sqrt{e^{4x}-1}}\cdot e^{4x}\cdot 4}{(\sqrt{e^{4x}-1})^2} = \frac{4e^{4x}\sqrt{e^{4x}-1} - \dfrac{2e^{8x}}{\sqrt{e^{4x}-1}}}{e^{4x}-1}$$

$$= \frac{4e^{4x}(e^{4x}-1) - 2e^{8x}}{(e^{4x}-1)\sqrt{e^{4x}-1}} = \frac{2e^{4x}[2(e^{4x}-1) - e^{4x}]}{(e^{4x}-1)^{\frac{3}{2}}} = \frac{2e^{4x}(e^{4x}-2)}{(e^{4x}-1)^{\frac{3}{2}}}$$

[5] (c) $f(x) = (1 + 2\ln x)^4 \ln^2 x$

$$f'(x) = (1+2\ln x)^4 \cdot 2\ln x \cdot \frac{1}{x} + \ln^2 x \cdot 4(1+2\ln x)^3 \cdot \frac{2}{x}$$

$$= \frac{2\ln(x)\,(1+2\ln x)^3[(1+2\ln x) + 4\ln x]}{x}$$

$$= \frac{2\ln x\,(1+2\ln x)^3(1+6\ln x)}{x}$$

[5] (d) $f(x) = (\sin x)^{\sin x}$

$$\ln f(x) = \sin x\,\ln(\sin x)$$

$$\frac{1}{f(x)}f'(x) = \sin x \cdot \frac{1}{\sin x}\cdot \cos x + \ln(\sin x)\cdot \cos x = \cos x + \cos x\,\ln(\sin x)$$

$$= \cos x(1 + \ln(\sin x))$$

$$f'(x) = f(x)[\cos x(1+\ln(\sin x))] = (\sin x)^{\sin x}\cos x\,(1+\ln(\sin x))$$

[5] 6. (a) Find and simplify the derivative of

$$f(x) = x\tan^{-1}\left(\frac{x}{4}\right) - 2\ln(x^2+16) \qquad \left[\text{ Note that } \tan^{-1}x = \arctan x\ \right]$$

$$f'(x) = x \cdot \frac{1}{1+(\frac{x}{4})^2} \cdot \frac{1}{4} + \tan^{-1}\left(\frac{x}{4}\right) - 2 \cdot \frac{1}{x^2+16} \cdot 2x$$

$$= \frac{4x}{16+x^2} + \tan^{-1}\left(\frac{x}{4}\right) - \frac{4x}{x^2+16} = \tan^{-1}\left(\frac{x}{4}\right)$$

[5] (b) Find and simplify the second derivative y'' of the function given by

$$y = \sinh^2 3x$$

$$y' = 2\sinh 3x \cosh 3x \cdot 3 = 6\sinh 3x \cosh 3x$$

$$y'' = 6\left[\sinh 3x \sinh 3x \cdot 3 + \cosh 3x \cosh 3x \cdot 3\right] = 6(3\sinh^2 3x + 3\cosh^2 3x)$$

$$= 18\left(\sinh^2 3x + \cosh^2 3x\right)$$

7. Find each of the following limits:

[4] (a) $\lim\limits_{x\to 0} \dfrac{\cos x - \cos 3x}{x^2}$

$$\lim_{x\to 0} \frac{\cos x - \cos 3x}{x^2} = \lim_{x\to 0} \frac{-\sin x + 3\sin 3x}{2x} = \lim_{x\to 0} \frac{-\cos x + 9\cos 3x}{2} = \frac{-1+9}{2} = 4$$

[5] (b) $\lim\limits_{x\to 0^+} (1+2\sin x)^{\frac{2}{x}}$

$\lim\limits_{x\to 0^+} (1+2\sin x)^{\frac{2}{x}}$ (1^∞ type) Let $y = \lim\limits_{x\to 0^+} (1+2\sin x)^{\frac{2}{x}}$

$$\ln y = \lim_{x\to 0^+} \frac{2}{x}\ln(1+2\sin x) = \lim_{x\to 0^+} \frac{2\ln(1+2\sin x)}{x} = \lim_{x\to 0^+} \frac{2 \cdot \dfrac{1}{1+2\sin x} \cdot 2\cos x}{1}$$

$$= \lim_{x\to 0^+} \frac{4\cos x}{1+2\sin x} = \frac{4}{1+0} = 4$$

So $y = e^4$. Thus $\lim\limits_{x\to 0^+} (1+2\sin x)^{\frac{2}{x}} = e^4$

[5] 8. Find an equation of the normal line to the graph of the equation

$$2x - y\ln y = 4$$

at the point $(2,1)$.

$$2x - y\ln y = 4$$

$$2 - \left(y \cdot \frac{1}{y} \cdot y' + \ln y \cdot y'\right) = 0$$

$$2 - (1 + \ln y)y' = 0$$

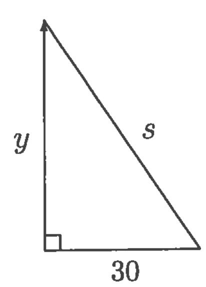
$$y' = \frac{2}{1+\ln y}$$

At $(2,1)$: $y' = \dfrac{2}{1+0} = 2$

$m_N = -\dfrac{1}{y'} = -\dfrac{1}{2}$ Equation : $y - 1 = -\frac{1}{2}(x-2)$

$y = -\frac{1}{2}x + 2$

[5] 9. A helicopter leaves the ground at a point 30 metres horizontally away from an observer and rises vertically at a rate of 2 m/sec. At what rate is the distance between the observer and the helicopter changing 20 seconds after the helicopter leaves the ground?



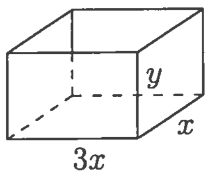
$s^2 = y^2 + 30^2$ Given $\dfrac{dy}{dt} = 2$, find $\dfrac{ds}{dt}$ when $t = 20$.

$$2s\frac{ds}{dt} = 2y\frac{dy}{dt} + 0 \quad \Rightarrow \quad \frac{ds}{dt} = \frac{y}{s}\frac{dy}{dt} = \frac{y}{s}(2) = \frac{2y}{s}$$

When $t = 20$, $y = 2(20) = 40$ and $s = \sqrt{40^2 + 30^2} = 50$. Then

$$\frac{ds}{dt} = \frac{2y}{s} = \frac{2(40)}{50} = \frac{8}{5} \text{ m/sec}$$

[8] 10. A closed rectangular box is to have base with length three times its width. Find the dimensions of the box of least surface area if the volume is to be 288 cm^3.



Let $x =$ width
$3x =$ length
$y =$ height
$S =$ surface area

We want to minimize

$$S = 2(3x^2) + 2(xy) + 2(3x)(y) = 6x^2 + 8xy$$

with $3x^2 y = 288$

$$y = \frac{96}{x^2}$$

$$S(x) = 6x^2 + 8x\left(\frac{96}{x^2}\right) = 6x^2 + \frac{768}{x}$$

$$S'(x) = 12x - \frac{768}{x^2} = \frac{12x^3 - 768}{x^2} = \frac{12(x^3 - 64)}{x^2}$$

$S'(x) = 0$ when $x = 4$

$S''(x) = 12 + \frac{1536}{x^3} \Rightarrow S''(5) = 12 + \frac{1536}{4^3} > 0$. So S has a minimum at $x = 4$. Then

$y = \frac{96}{16} = 6.$

So the dimensions should be

$$4 \text{ cm} \times 12 \text{ cm} \times 6 \text{ cm}.$$

[10] 11. Sketch the graph of $y = \frac{x^2}{(x+2)^2}$, giving intercepts, asymptotes, where increasing and where decreasing, any relative maximum and relative minimum points, where concave upward, where concave downward, and any inflection points. $\left[\text{Note: } y' = \frac{4x}{(x+2)^3} \text{ and } y'' = \frac{8(1-x)}{(x+2)^4}\right]$

$$y = \frac{x^2}{(x+2)^2}, \quad y' = \frac{4x}{(x+2)^3}, \quad y'' = \frac{8(1-x)}{(x+2)^4}$$

Intercepts: $(0,0)$

Asymptotes: $y = 1$, $x = -2$

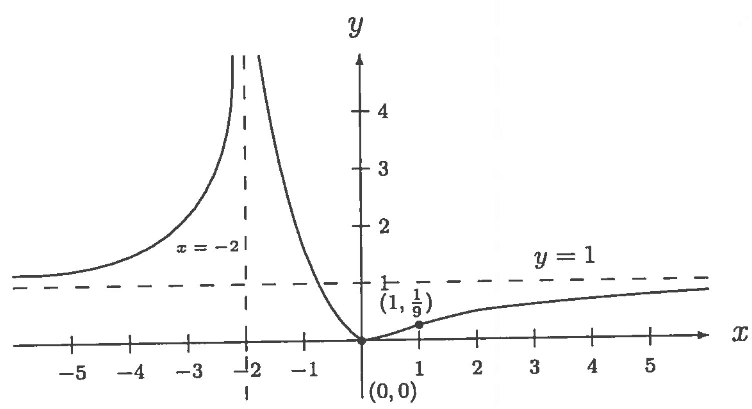
Other:

$y' = 0$ for $x = 0$
$y' = \infty$ for $x = -2$

$-\infty < x < -2$	$y' > 0$	increasing	
$-2 < x < 0$	$y' < 0$	decreasing	$(0,0)$ rel. min.
$0 < x < \infty$	$y' > 0$	increasing	

$y'' = 0$ for $x = 1$
$y'' = \infty$ for $x = -2$

$-\infty < x < -2$	$y'' > 0$	concave up	
$-2 < x < 1$	$y'' > 0$	concave up	$(1, \frac{1}{9})$ inflection point
$1 < x < \infty$	$y'' < 0$	concave down	



[5] 12. Answer **one** of (a) or (b): Determine, with reasons, whether the given statement is True or False. No marks will be given for a correct answer without a valid justification.

100

(a) $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for all positive integers n.

(b) The function $f(x) = |x-2|$ is differentiable at $x = 2$.

(a) Using L'Hospital's Rule we get

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \quad \text{etc.}$$

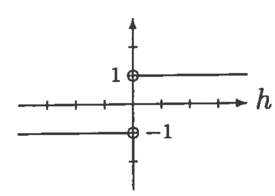
Then applying L'Hospital's Rule n times we get

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{0}{e^x} = 0$$

So the statement is **True**.

(b) $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{|(2+h)-2|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$

does not exist since $\lim_{h\to 0^-} \frac{|h|}{h} = -1$ and $\lim_{h\to 0^+} \frac{|h|}{h} = 1$.



So f is not differentiable at $x = 2$, and the statement is **False**.