LAST QUESTION ON FINAL EXAM

EXAMPLES

Last question often most challenging but not weighted heavily in exam, usually offers choice, partial marks given, and similar to assignment extended concept questions, and partial marks given – so relax and enjoy ! Also, obviously shouldn't be the main focus of your studying.

<u>Proof Examples</u>

Ex. 1

Prove that if $f(x)$ and $g(x)$ are differentiable functions then $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

Ex. 2 For $f(x) = \arctan x$, prove that $f'(x) = \dfrac{1}{x^2 + 1}$

Ex. 3 Using the derivative rules and your knowledge of the log rules, prove that

$$\frac{d}{dx}\left(\ln[f(x)\,g(x)] \right) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}$$

Ex. 4 Using the definition of the derivative, prove that $[\cos x]' = -\sin x$

The following identities may (or not !) be helpful:

$\cos(a+b) = \cos a \cos b - \sin a \sin b$, $\cos(a-b) = \cos a \cos b + \sin a \sin b$

Ex. 5 Using whatever method you like, prove that $(\tan x)' = \sec^2 x$

Ex. 6 Use the definition of the derivative to find $f'(x)$ for $f(x) = e^x$

<u>Extension of Concept(s)</u>

Ex. 7 Sketch the graph of a function $f(x)$ that satisfies all the following conditions:

$f(0) = f(3) = f(7) = 0$

$\lim_{x \to +\infty} f(x) = 0$, $\lim_{x \to 6} f(x) = -\infty$

$f'(-2) = f'(1) = f'(9) = 0$

$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,

$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,

$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,

$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$

Ex. 8 For the function $f(x)$ given below with its derivatives, find values of constants a and b which make all of the following true:

i) $f(x)$ has a horizontal asymptote at $y = -1$

ii) $f(x)$ has critical points at $x = 0, -4$

iii) $f(x)$ has flex points at $x = 0, 1$

$$f(x) = \frac{ax^2 - bx + c}{x^2}, \quad f'(x) = \frac{bx(x+2c)}{x^4}, \quad f''(x) = \frac{(2bx - 5c)}{x^4}$$

Ex. 9

Not applicable

Ex. 10

If $f(x)$ is nonzero, differentiable and increasing on the open interval (a, b), show that the function $g(x) = \dfrac{1}{f(x)}$ is decreasing on (a, b).

More Unusual Topics

Ex. 11

Not Applicable

Mixed Skills

Limits combining various types of limits, problems involving more complicated simplification, etc..

Ex. 13 Given that $xy + y^2 = 1$, show that $y'' = \dfrac{2}{(x+2y)^3}$.

FINAL QUESTIONS EXAMPLES SOLUTIONS

PROOF EXAMPLES

Ex. 1

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

TRICK +/- TERM

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}$$

$$= \lim_{h\to 0} \left\{ \underbrace{\left[\frac{f(x+h) - f(x)}{h}\right]}_{f'(x)} \underbrace{g(x+h)}_{g(x)} + f(x) \underbrace{\left[\frac{g(x+h) - g(x)}{h}\right]}_{g'(x)} \right\}$$

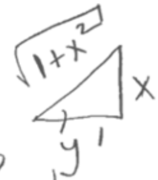
$$= f'(x)g(x) + f(x)g'(x)$$

$$\therefore \underline{[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)}$$

Ex 2

$y = \text{Arctan}\, x$

$x = \tan y \implies \dfrac{x}{1} = \tan y$



$\dfrac{d}{dx} x = \dfrac{d}{dx} \tan y$

$1 = \sec^2 y \; y'$

$y' = \dfrac{1}{\sec^2 y}$

$y' = \cos^2 y = \dfrac{1}{\left(\sqrt{1+x^2}\right)^2} = \dfrac{1}{1+x^2}$

$\therefore\ y' = \dfrac{1}{1+x^2}$

Ex 3

$\dfrac{d}{dx}\left[\ln[f(x)g(x)]\right]$

$= \dfrac{d}{dx}\left[\ln f(x) + \ln g(x)\right]$ As $\ln ab = \ln a + \ln b$

$= [\ln f(x)]' + [\ln g(x)]'$

$= \dfrac{1}{f(x)} f'(x) + \dfrac{1}{g(x)} g'(x)$ } By Chain Rule

$= \dfrac{f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)}$

Ex#

$f(x) = \cos x$

$f(x+h) = \cos(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

BECAUSE $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \left[\cos x \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} - \sin x \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right]$$

$$= -\sin x$$

$\therefore \; (\cos x)' = -\sin x$

Ex 5

$(\tan x)'$

$= \left(\dfrac{\sin x}{\cos x}\right)'$

$= \dfrac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$ By Quot. Rule

$= \dfrac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$

$= \sec^2 x$

$\therefore (\tan x)' = \sec^2 x$

Ex. 6

$f(x) = e^x$

$f(x+h) = e^{(x+h)}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$ Because $e^a e^b = e^{(a+b)}$

$$= \lim_{h \to 0} e^x \frac{\cancel{(e^h - 1)}}{\cancel{h}}_{\,1}$$ As $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$= \underline{\underline{e^x}}$$

EXTENSION OF CONCEPTS

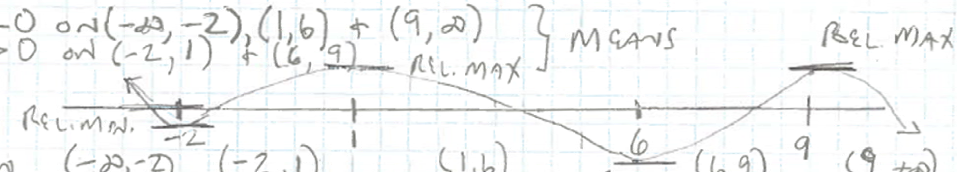
Ex. 7

① $f(0)=f(3)=f(7)=0$ MEANS
X INT. : $(0,0), (3,0), (7,0)$ Y INT. $(0,0)$

② $\lim_{x\to+\infty} f(x)=0$ MEANS HA AT $y=0$.
$\lim_{x\to 6} f(x)=-\infty$ MEANS VA AT $x=6$
& MEANS $\lim_{x\to 6^-} f(x)=\lim_{x\to 6^+} f(x)=-\infty$

③ $f'(-2)=f'(1)=f'(9)=0$ MEANS CPs AT $x=-2,1,9$

④ $f'(x)<0$ ON $(-\infty,-2),(1,6)$ & $(9,\infty)$
$f'(x)>0$ ON $(-2,1)$ & $(6,9)$ } MEANS



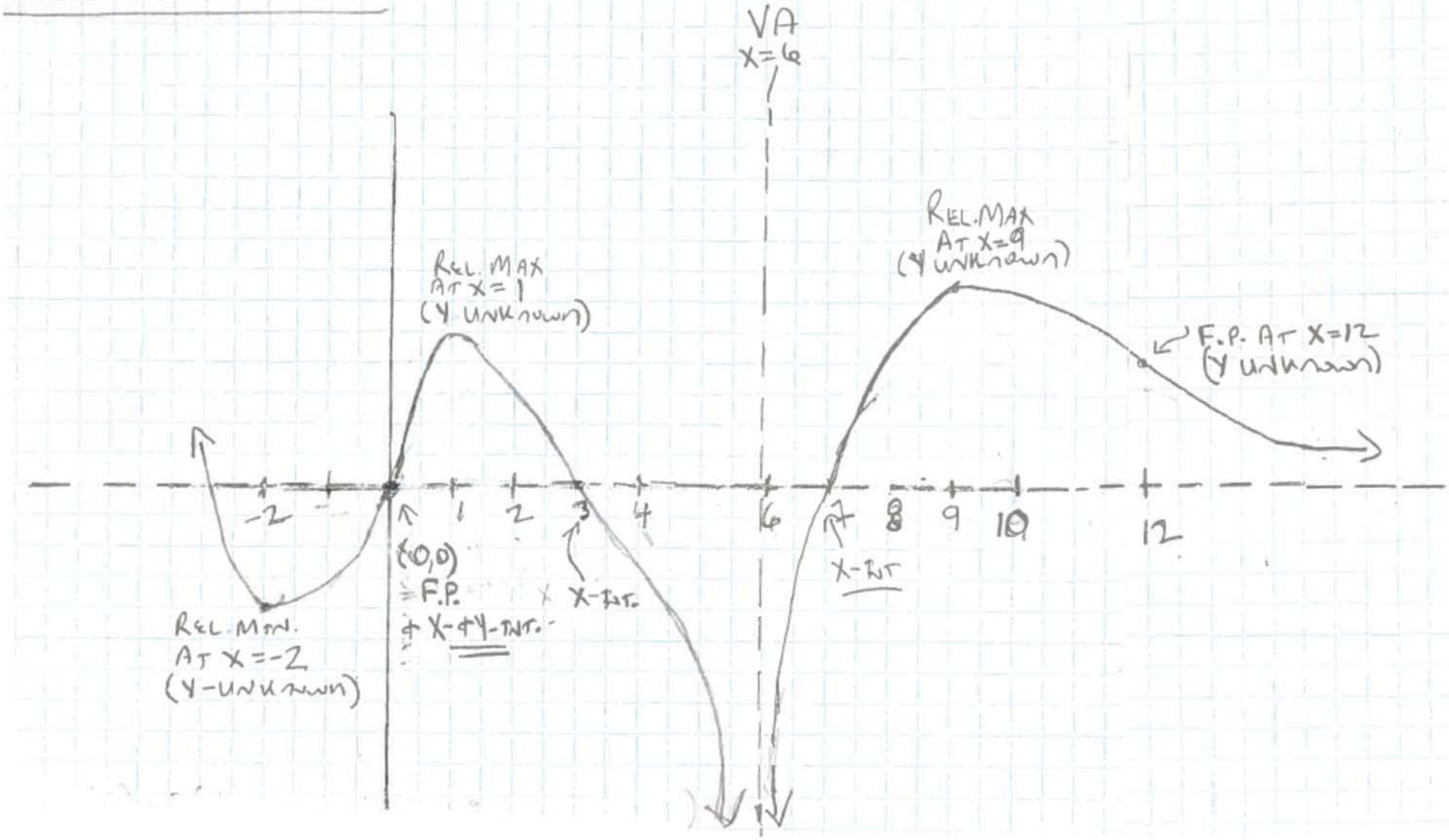
REGION	$(-\infty,-2)$	$(-2,1)$	$(1,6)$	$(6,9)$	$(9,+\infty)$
	$x<-2$	$-2<x<1$	$1<x<6$	$6<x<9$	$x>9$
$f'(x)$	⊖	⊕	⊖	⊕	⊖
$f(x)$	↓	↑	↓	↑	↓

⑤ $f''(x)>0$ ON $(-\infty,0)$ & $(12,+\infty)$, $f''(x)<0$ ON $(0,6)$ & $(6,12)$
MEANS

∪ F.P. ∩ NOT FP ∩ F.P. ∪

REGIONS	$(-\infty,0)$	0	$(0,6)$	6	$(6,12)$	12	$(12,+\infty)$
	$=x<0$		$=0<x<6$		$=6<x<12$		$=x>12$
$f''(x)$	⊕		⊖		⊖		⊕
$f(x)$	∪		∩		∩		∪

SION OF CONCEPTS



Ex. 8

(i) $f(x)$ HAS HA AT $y=-1$

HA: $y = \lim_{x \to \pm\infty} f(x)$

$\therefore -1 = \lim_{x \to \pm\infty} \dfrac{ax^2 - bx + c}{(1)x^2}$

$\Rightarrow -1 = \dfrac{a}{1}$ AS POWER TOP = BOTTOM

$\Rightarrow \underline{\underline{a = -1}}$

(ii) $f(x)$ HAS C.Ps AT $x = 0, -4$*

CPs: $f'(x) = 0, \dfrac{n}{0}$

$f'(x) = \dfrac{bx(x+2c)}{x^4}$

$bx(x+2c) = 0$ OR $x^4 = 0$

$bx = 0$ OR $x + 2c = 0$

$\therefore x = 0$ $\quad x = -2c$

*$\therefore -4 = -2c$

$\therefore c = \dfrac{-4}{-2}$

$\therefore \underline{\underline{c = 2}}$

(iii) $f(x)$ HAS FPs AT $x = 0, 1$

FPs: $f''(x) = 0, n/0$

$f''(x) = \dfrac{(2bx - 5c)}{x^4}$

$2bx - 5c = 0$ $\qquad x^4 = 0$

$2bx = 5c$ $\qquad \underline{\underline{x = 0}}$

$2b(1) = 5(2)$ FROM (ii)

FROM FP AT $x=1$

$2b = 10$

$\underline{\underline{b = 5}}$

$\therefore \underline{\underline{a = -1,\ b = 5,\ c = 2}}$

Ex. 10

Since $f(x)$ is increasing on (a, b), $f'(x) > 0$ on (a, b), and then, since $f(x) \neq 0$ on (a, b) it follows that

$$g'(x) = -\frac{f'(x)}{[f(x)]^2} < 0$$

on (a, b) since $[f(x)]^2 > 0$. So $g(x)$ is decreasing on (a, b).

Ex. 11

Not Applicable

Ex. 12

$$y' = \frac{-y}{x+2y}$$

$$y'' = -\left[\frac{y'(x+2y) - y\overbrace{(x+2y)'}^{1+2y'}}{(x+2y)^2}\right]$$

$$= -\left[\frac{y'x + \cancel{2yy'} - y - \cancel{2yy'}}{(x+2y)^2}\right]$$

$$= -\left[\frac{y'x - y}{(x+2y)^2}\right]$$

$$= -\left[\frac{\left(\frac{-y}{x+2y}\right)x - y}{(x+2y)^2}\right]$$

$$= -\left[\frac{-yx - yx - 2y^2}{(x+2y)^3}\right]$$

$$= -\left[\frac{-2yx - 2y^2}{(x+2y)^3}\right]$$

$$= \cancel{-\left[\frac{-2y(x+y)}{(x+2y)^3}\right]}$$

$$= +\left[\frac{2\cancel{(xy+y^2)}^{1*}}{(x+2y)^3}\right]$$

$$= \frac{2}{(x+2y)^3}$$

* AS ORIGINAL FUNCTION $xy + y^2 = 1$